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A Computational Concept for Normative Equity*,†

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Abstract

The relative poverty concept of the European Union leads to a new approach for measuring equity and inequity in societies. The approach results from a differential equation for a one-parametric class of Lorenz curves. These allow to express societal inequity in terms of a so-called equity parameter. This parameter is not only of statistical or other, large-scale descriptive nature but it relates individual welfare to overall equity. The paper deals with the mathematical aspects, fitting questions, empirical findings, consequences for the middle class and some consequences for future growth processes aiming at more global equity and a sustainable development.

Keywords: factor 10, income distribution, index numbers and aggregation, sustainable development, world contract

JEL Classification: O15, C430, Q01

1. Introduction

This paper reflects considerations within the Information Society Forum of the European Commission and its Global Society Dialogue and of the European Eco-Social Forum on development issues within a global information society, (cf. Friewald-Hofbauer and Scheiber, 2001; Greiner and Radermacher, 1994; Information Society Forum, 2000; Radermacher, 1999, 2000; Schauer, 2000; Schauer and Radermacher, 2001; Strohm, 2000; van Dijk, Pestel, and Radermacher, 1999). Particularly with a view to reaching sustainability within the present globalization processes, the crucial role of achieving a higher level of equity world-wide is considered to be essential. A much higher level of equity is needed to find consensus between states for what is called a double-factor-10 concept. This concept combines tenfold growth with tenfold eco-efficiency, but also requires to limit use of critical

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resources and observance of critical environmental limits strictly within present levels to allow a sustainable transition into the future.

With a view to such a factor-10 frame, which offers many new opportunities, more equity is the key to world contracts within a consensus model to strictly limit environmental burdens within a concept of co-funding of development, as is typical in the more local globalization process of extending the European Union. The issue in all its many system-theoretical as well as political considerations and consequences, including the relationship with new patterns in population development, is elaborated in separate papers by the third author. The present more theoretical and, in part, mathematical work is the methodological underpinning of these other considerations, which have been the starting point of this work, (cf. Radermacher, 2001, 2002).

With this aim in mind, the present paper in Section 2 gives a new mathematical approach to welfare equity. Welfare is typically measured by income or consumption with its distribution over the population being described by the Lorenz curve. We investigate a special class of Lorenz curves which result from a solution of a linear inhomogeneous differential equation. The differential equation depends on some parameter $\varepsilon = 1/a$, where ε is called the equity parameter. This approach is related to the Gini index and other measures of social (in-)equality. However, it allows a more individual interpretation, results directly from the EU definition of poverty, is helpful for guiding policy measures and gives a powerful instrument for modeling and simulation of future developments, depending on policy measures. More information on such issues is given in Section 2.4.

In Section 3, aspects of fitting Lorenz curves to data from the Worldbank and empirical findings concerning this new concept of equity are given. Typical EU-equity factors lie between 1 : 1.54 (65%) and 1 : 1.85 (54%), with Great Britain close to 1:2 and Portugal at exactly 1 : 2 (50%) being an exception and just within the limits of the EU poverty definition, while the equity in the US is only 1 : 2.14 (47%).

Section 4 deals with merging of populations. This leads to a better understanding of what happens to the equity factor if countries with different levels of per capita average income and different equity factors are viewed together, as is typical e.g. for the European unification process as for studying the world as a whole.

Section 5 is about growth and inequity reduction and related issues. While an equity factor 1 : 1 (100%) (extreme communism) historically did not work, high degrees of inequity (beyond 1 : 3.5 (28%)) can obviously also not work and are characteristic for certain colonial and/or apartheid regimes. This has to do with the observation that under such conditions huge parts of society cannot be really value creating but will be occupied as personal servants. This is further developed in Section 5.2. It seems that all economically well-performing nations are found in a quite limited range of equity (factors between 1 : 1.5 (66%) and 1 : 2.2 (45%). In this context we can also give further indications, playing with the absolute and relative situation of almost all people (reflecting Rawls' theory of justice), why most developed countries do not "voluntarily" opt for more inequity into the direction of the US model, independent of what their present equity level is.

The paper then studies extensively a situation which corresponds to the actual global picture, namely a situation where 20 percent of the world population own 80 percent of world income and world resources, while the other 80 percent of the population have to get along with 20 percent. The paper develops methods to determine a lower bound on the equity factor in this case. It turns out that the worlds equity state is worse than 1 : 8 (12.5%), showing a picture of global apartheid, certainly one of the reasons for the recent tensions with globalization processes around the world. The paper also gives some hints how, within a factor-10 concept, growth rates over the next 50 years might reasonably be split between the North and the South (Radermacher, 2001), with maybe a factor 4 for the North and a factor 34 for the South (10 → 4: 34 formula). This corresponds to reasonable growth rates for richer as well as developing countries and would allow a solid and robust route to a more balanced global situation consistent with a sustainable development.

Also, the observation from Section 5.4 are interesting. First, we see that reaching the average income of 1 is a monotonic function of the equity parameter $\varepsilon = 1/a$; starting with 0.6651 for $a = 1.2$ and leading to 0.8425 for $a = 4.0$. If we define the middle class as the population segment [60%, 90%], then this group starts losing relatively for a beyond 1.6, while the lower middle class [60%, 80%] starts losing relatively already for a beyond 1.2 and the upper middle class [80%, 90%] for $a = 2.0$.

We conclude with an outlook in Section 6 setting results into perspective.

2. Approach

2.1. *Definitions and basic properties*

Among the class of all Lorenz curves F, the focus here is on a particular curve type, that satisfies the ordinary differential equation

$$F'(x) = \frac{1}{a} \cdot \frac{1 - F(x)}{1 - x}$$

for $x \in [0, 1]$ and some real-valued parameter $a > 1$, where $\varepsilon = 1/a$ is called the equity parameter. The normalization conditions $F(0) = 0$ and $F(1) = 1$ must also be satisfied. The normalization accounts for relative welfare.

The rationale behind the differential equation is as follows. Quantile x of the population receives its proportion of welfare. The remaining welfare $1 - F(x)$ is averaged over the remaining population $1 - x$. The poorest remaining receives fraction $\varepsilon = 1/a$ thereof, in complete analogy to the overall poorest receiving fraction $\varepsilon = 1/a$ of the overall average. The relation between the poorest to all the richer is thus homogeneous over the complete scale. Since the density of the welfare distribution is approximately proportional to individuals' welfare shares, the fraction $\frac{1}{a} \cdot \frac{1-F(x)}{1-x}$ equals the derivative of the unknown Lorenz curve.

The approach is motivated by a relative poverty notion, (cf. European Parliament, 1999; Finland), where poverty means that the overall poorest falls short of 50% of the overall per capita income. Attaining that poverty bound, without shortfall, amounts to $a = 2$ or $\varepsilon = 1/2$ (50%) for $x = 0$. The rationale of the differential equation is to extend this view to any population segment $[x, 1]$, and to admit other equity parameters than $\varepsilon = 1/2$ (50%). The differential equation is linear inhomogeneous with general solution

$$F_a(x) = 1 - (1 - x)^{\frac{1}{a}}$$

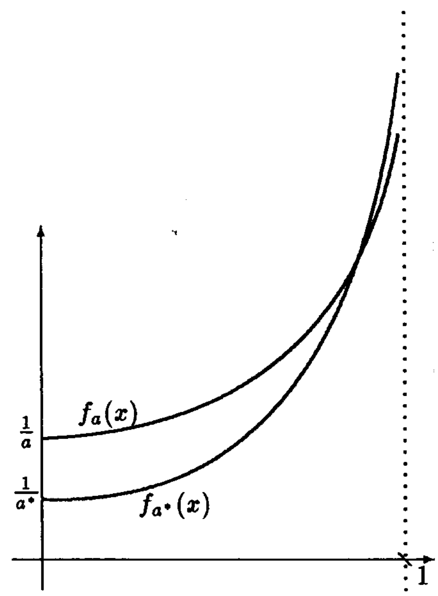

Figure 1. Sketch of two densities $f_a(x)$ and $f_{a^*}(x)$ for $a^* > a$.

resp.

$$F'_a(x) = f_a(x) = \frac{1}{a} \cdot (1-x)^{\frac{1}{a}-1} \quad \text{with } f_a(0) = \frac{1}{a}.$$

Functions F_a and f_a are increasing and strictly convex, since the respective first and second derivatives are positive. All densities f_a have the same asymptotic line at $x = 1$. The larger a is, the "less uniform" the density, (cf. figure 1). The area under the Lorenz curve is $\int_0^1 F_a(x)\,dx = 1/(a+1)$, which is decreasing in parameter a, so that the expectation $\int_0^1 x f_a(x)\,dx = a/(a+1)$ is increasing. In principle, either quantity suffices to determine the equity parameter $\varepsilon = 1/a$.

While the densities intersect exactly once for different equity parameters, the Lorenz curves themselves do not intersect anywhere between the endpoints. This is called Lorenz dominance, see for example Litchfield (1999). The upper Lorenz curve (belonging to the smaller equity parameter) dominates the lower curve. Noteworthy, every Lorenz curve satisfies the inequality $F(x) \leq x$ for all $0 \leq x \leq 1$.

The limit $\lim_{a \to 1+} F_a(x) = x$ describes the egalitarian or uniform distribution. In a uniformly distributed population all individuals are equally wealthy—or equally poor. This corresponds to an extreme communist situation. However, to all historical experience, this means an equal share for all on an absolutely low level. The equal distribution is often used as the reference for other distributions and sometimes it is even attributed as fair (Leslie, 2000). The Gini index measures the inequality of a welfare distribution in terms of integral distance from the egalitarian distribution. This index can be computed here in closed form as

$$\int_0^1 x - F_a(x)\,dx = \frac{a}{a+1} - \frac{1}{2}$$

which is increasing in parameter a. The difference between the expectation and the Gini index is 0.5 for all Lorenz curves. The Gini index belonging to a curve of type F_a is called equity Gini index. Occasionally, and sometimes without warning, as obvious in Worldbank (2001), the Gini index of an arbitrary Lorenz curve F receives a factor of 2, then being defined by $2\int_0^1 x - F(x)\,dx$, (cf. Leslie, 2000). The possible Gini indices then range between 0 and 1, instead of 0 and .5.

It should be pointed out that there is no ideal or theoretically optimal value for the equity parameter. Neither values that are too large nor values that are too small seem to be acceptable from a society point of view for all historical experience as well as from some straightforward economic considerations. This is in analogy to the Gini index itself and other inequality measures for which no ideal value can be set.

Major use of the equity parameter and approximations thereof will be made in form of ordinal comparisons between welfare distributions. The original reason for introducing the equity parameter is to relate individuals' welfare situations with welfare distributions. There we build on the impressively "intelligent" relative poverty concept of the EU. This approach will allow to go beyond the descriptive character of Lorenz curves by some normative and pragmatic views. This supports intuition for "understanding" Lorenz curves as well as arguments for societal redistribution.

2.2. *Further properties*

Lorenz curves of type F_a have several interesting properties. One is the iteration property $F_{c\cdot a}(x) = F_c(F_a(x)) = F_a(F_c(x))\ \forall a \geq 1\ \forall c \geq 1$. Also, for integers $a = 1, 2 \ldots$, function F_a is the probability distribution function of each of stochastically independent random variables whose minimum is uniformly distributed. Formally, the latter means $P(\min\{X_1, \ldots, X_a\} > x) = P(U > x)$ for $P(U \leq x) = x$ and $P(X_i \leq x) = F_a(x)$ for all $i = 1, \ldots, a$.

2.3. *Related work*

The present welfare distribution functions differ from β-distributions with $F(x) = x - \vartheta x^{\gamma}(1-x)^{\delta}$ and general quadratic welfare distributions $F(x) = \frac{1}{2}(bx + e + \sqrt{mx^2 + nx + e^2})$, which appear to be motivated empirically (Datt, 1998). The current type of Lorenz curve has earlier been proposed by Rasche et al. (1980), even in the more general form $F(x) = (1-(1-x)^{\alpha})^{\frac{1}{\beta}}$ with $0 < \alpha < 1$ and $\beta \leq 1$. This curve type appears to have been motivated by inappropriate curvature features of other candidates for parametric Lorenz curves.

Also, a variation of the Rasche curves like $F(x) = x^{\alpha}(1-(1-x)^{\beta})$, $a > 0, 0 < \beta \leq 1$, and exponential curves such as $F(x) = \frac{e^{\kappa x}-1}{e^{\kappa}-1}$, $\kappa > 0$, have been investigated, see Cheong (2002). A survey on parametric Lorenz curves, with emphasis on β- and similar distributions, is provided in Chotikapanich and Griffiths (1999). The function types of curves seem to be adapted from statistics, and then investigated for their goodness of fit, rather than derived from a *principle*.

Parametric Lorenz curves are occasionally complemented by non-parametric methods, such as kernel estimation and quantile ratios (Ginther, 1995). The lack of a-priori justification for a certain type of curve and simplicity issues motivate such approaches. An overview on inequity measurements can be found in the edition (Silber, 1999).

Formal poverty measures, like poverty lines, poverty gaps (degree of shortfall of poverty line), Sen's index, the Lorenz family of inequality measures (Aaberge, 2000), and the Foster-Greer-Thorbecke FGT measures (Foster, Greer, and Thorbecke, 1984) are not considered here even if tuned to relative poverty concepts (Schwartzman, 1998). Poverty measures typically focus on lower segments of Lorenz curves, thus letting poverty appear as a *boundary* problem of welfare inequality. This view is not adopted here; rather, poverty is an issue of the complete distribution.

Inequality indices requiring other than income or consumption data such as generalized entropy measures and the Atkinson index, see for example Litchfield (1999), are not considered. The reason is that all inequality information should be obtainable without external parametrization.

2.4. *Ten advantages of the new equity measure*

The principal message of the new measure is not too different from other (in-)equality measures. Particularly with regard to the Gini coefficient, the ranking of the states of the world according to the degree of inequality shows a similar picture. The reason for possible differences is that the same Gini coefficient may belong to different distribution schemes (Lorenz functions), see end of Section 3.2. Note, once again, that the equity parameter ε relates to precisely one Lorenz curve, namely F_ε.

1. The equity parameter ε is similar in content as the Gini index and other measures, but relates it to one, precise Lorenz curve, F_ε, which has a number of special properties.
2. The Lorenz curve F_ε allows an interpretation with respect to individuals, e.g. it indicates the (relative) income of the overall poorest people and of the poorest people within any segment $[x, 1]$ of richest people. This can be used directly and communicated in politics, cf. the poverty definition of EU.
3. The relative wealth concept is used also in other contexts of EU policies. For instance, EU structural funds are allocated to regions of objective I status which means that their people's average income is 75% or less of the EU average, (cf. Norway, 2000).
4. Point 2 reflects an assumed self-similarity over the whole society. By this, ε, as a best fit over all $x \in [0, 1]$ gets a higher robustness (against "outliers") compared to only considering the lowest income in the society.
5. The self-similarity assumption in 4 reflects the idea of a pyramidal structure of a society. High incomes mean the right of access to huge amount of working time of people with lower income and/or lower service qualification, which admit a certain access to services of people with even lower income and qualification and so on. Consequently, in societies with high equity factor, there only are few personal services, because there

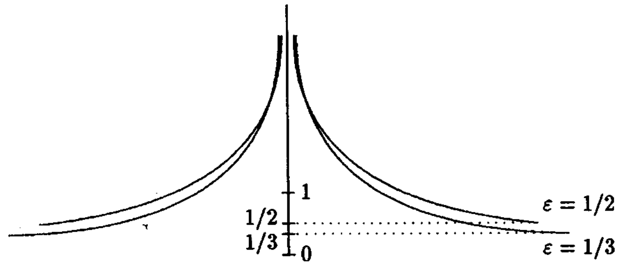

Figure 2. Sketch of pyramidal arrangement of income densities for the parameters $\varepsilon = 1/2$ and $\varepsilon = 1/3$.

is only little potential to pay for the time of others. Figure 2 shows, in the form of a pyramidal structure, the relative volume of people on a particular income level, relative to an average income of 1. Note the pyramidal structure of a society that reflects access to service times and qualities.

The distribution of income, which is not normalized to population segments, can be considered as a random variable X. Its distribution function can be computed from the density of the Lorenz curve by the identity

$$P(X \leq f_a(x)) = x, 0 < x < 1.$$

The income distribution function is then given from the inverse of the derivative of the Lorenz curve by

$$\Phi_a(x) := P(X \leq x) = f_a^{-1}(x) = 1 - (ax)^{a/(1-a)}, \frac{1}{a} \leq x.$$

The density of the income distribution then is

$$\varphi_a(x) = \Phi_a'(x) = \frac{a}{a-1} \cdot \frac{1}{a^{a/(a-1)}} \cdot \frac{1}{x^{a/(a-1)+1}}, \frac{1}{a} \leq x.$$

This transformation is the inverse of the well known transformation from income distributions to Lorenz curves. That transformation, see for example Leslie (2000), is given for general distributions by

$$F(x) = \frac{\int_0^{\Phi^{-1}(x)} u\varphi(u)\,du}{\int_0^{\infty} u\varphi(u)\,du}, \quad 0 < x < 1,$$

where $\varphi(x)$ and $\Phi(x) = \int_0^x \varphi(u)\,du$ denote the density and distribution function respectively of the income variable. This transformation applied to the particular functions $\varphi_a(x)$ and $\Phi_a(x)$ indeed reproduces the Lorenz curves $F_a(x)$. All densities here have unit mean, i.e. $\int_0^{\infty} u\varphi_a(u)\,du = 1$. Sample income curves are given in Table 1.

Table 1. Sample income curves (densities and distribution functions).

$a = 1.5$	$\varphi_a(x) = \frac{8}{9} \cdot \frac{1}{x^4}$	$\Phi_a(x) = 1 - \frac{8}{27} \cdot \frac{1}{x^3}$	$\frac{2}{3} \le x < \infty$
$a = 2$	$\varphi_a(x) = \frac{1}{2} \cdot \frac{1}{x^3}$	$\Phi_a(x) = 1 - \frac{1}{4} \cdot \frac{1}{x^3}$	$\frac{1}{2} \le x < \infty$
$a = 3$	$\varphi_a(x) = \frac{1}{2\sqrt{3}} \cdot \frac{1}{x^{3/2}}$	$\Phi_a(x) = 1 - \frac{1}{\sqrt{27}} \cdot \frac{1}{x^{3/2}}$	$\frac{1}{3} \le x < \infty$

Any two densities with different parameters intersect illustrating that larger inequity corresponds to a larger frequency of high incomes. The same applies to any two distribution functions. Graphs of the income densities together with their images when reflected at the abscissa, are now rotated counterclockwise by 90° degrees. These then constitute "pyramids" as shown in figure 2.

6. All Lorenz curves F_ϵ fulfill a power law, as to be expected for $x \rightarrow 1$ by general considerations, comp. (Bouchard and Mezard, 2000). More precisely, doubling income, the fraction of the population with at least that income shrinks by a constant factor $\beta(\epsilon) = 2^{1/(\epsilon-1)}$. For $\epsilon = 1/2$ this factor becomes $\beta = 1/4$.
7. The equity factor ϵ taken as a measure of inequality, fulfills several of the general requirements for inequality measurements, such as the Pigou-Dalton transfer principle, income scale invariance, the population principle and the anonymity principle; see Litchfield (1999) for these general requirements.
8. With respect to worldwide sustainability and to overcoming poverty, the outline of a "landing platform 2025 and/or 2050" with time tables and performance criteria is an important topic. Within future scenario modelling, dealing with equity and related issues is a particularly difficult task. The present equity value is suited for such scenario analysis. It is a powerful instrument for operationalizing (in-)equity within simulation studies.
9. Considerable difficulties are caused by the equity assessment of an aggregated political entity, such as the EU, based on the equity values of its nations. We show in this paper how the new equity parameter allows to do this. Given the population numbers, total income and equity parameters of individual nations, the joint equity value can be approximately computed. This is done e.g. for the EU and to some extent for the world as a whole.
10. The $10 \rightarrow 4:34$ formula aiming at global sustainability within a 50–100 years time frame (Section 5.3), the investigations on the influence of the equity value on the middle class (Section 5.4) as well as the considerations concerning growth and (in-)equity as a parameter study (Section 5.5) can all only be done because the equity value is more than a number (as is the Gini index), but corresponds to a particular Lorenz function, that has nice properties but can also be handled analytically.

3. Fitting

Lorenz curves of type F_a will be fitted in order to approximate empirical welfare distributions. The approximations will exhibit features like Lorenz dominance which a pair of original distributions may not have. On the other hand, Lorenz dominance will be preserved by fitting.

The easiest fitting approach would compute the Gini index g of some (sampled) Lorenz curve and assess the equity parameter such that $g = a/(a+1)-0.5 \Leftrightarrow a = 1/(0.5-g)-1$. As this does not consider the shape of the original Lorenz curve, a regression approach is favoured instead even at the price of the Gini indices from the original curve and from the estimate being different; regressions need not be "Gini-consistent".

3.1. *Regression approach*

Any set of point values $(x_i, y_i), i = 1, \ldots, n$ of a welfare distribution gives rise to the one-parametric regression problem

$$\min_{a>1} \sum_{i=1}^{n} (F_a(x_i) - y_i)^2.$$

The quality of regression is exclusively measured here by the regression error $err = \sum_{i=1}^{n} (y_i - F_a(x_i))^2$. Since the optimal fitting problem has no closed solution, the parameter a will be ranged through a finite set of candidate values and the best is selected.

3.2. *Empirical findings*

Fitting was applied to data (x_i, y_i) which are given by the so-called world development indicators (Worldbank, 2001). That information is sparse, as only $n = 6$ data are given per distribution. These are all quintiles plus the highest and lowest decile. Data for different distributions may have been sampled in different years. This was ignored as the analysis is non-temporal. The underlying data were used as given without any quality test or modification. The data for India, which show a significant deviation from earlier data (not reported here), have been adjusted recently by the Worldbank to be better comparable to data of other nations (Chen, 2001).

Though some other statistics may provide more dense data in certain categories, see Luxembourg and Penn (2000) and US Bureau of Census (2001), these may be older or need normalization such as going from absolute to relative income. Also, not all data sources cover such a multitude of nations like the world development indicators. In addition, data were chosen to originate from one source since using data from different sources tends to make findings incomparable. A major source of that incomparability is (different) price adjustment of income data accounting for purchasing power.

Original data and best fit values for the inverse equity parameter a of 30 nations, found by regression over the domain $[1.01, 10.0]$ with step size 0.01, are summarized in Table 2.[1] These nations were selected from a set of more than 150 nations by a mixture of randomness and deliberation with a preference for Europe.

Explaining distributional inequality in terms of functions F_a is best obtained by choosing the stated values for the parameter a. One curve together with its sample points are sketched in figure 3. The equity parameters and the regression errors are summarized in Table 3. The errors are pairwise comparable, since all regressions are based on samples with identical support set.

Table 2. Income data and regression result.

Nation	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	x_6	y_6	a
Austria	0.1	0.044	0.2	0.104	0.4	0.252	0.6	0.437	0.8	0.666	0.9	0.807	1.54
Brazil	0.1	0.009	0.2	0.025	0.4	0.08	0.6	0.18	0.8	0.363	0.9	0.524	3.60
Canada	0.1	0.028	0.2	0.075	0.4	0.204	0.6	0.376	0.8	0.606	0.9	0.762	1.81
China	0.1	0.022	0.2	0.055	0.4	0.153	0.6	0.302	0.8	0.525	0.9	0.691	2.24
Czech Rep.	0.1	0.043	0.2	0.103	0.4	0.248	0.6	0.425	0.8	0.642	0.9	0.776	1.62
Denmark	0.1	0.036	0.2	0.096	0.4	0.245	0.6	0.428	0.8	0.655	0.9	0.795	1.59
Finland	0.1	0.042	0.2	0.1	0.4	0.242	0.6	0.418	0.8	0.641	0.9	0.784	1.63
France	0.1	0.028	0.2	0.072	0.4	0.198	0.6	0.37	0.8	0.598	0.9	0.749	1.86
Germany	0.1	0.037	0.2	0.09	0.4	0.225	0.6	0.4	0.8	0.629	0.9	0.774	1.70
Gr. Britain	0.1	0.026	0.2	0.066	0.4	0.181	0.6	0.344	0.8	0.571	0.9	0.727	1.99
Greece	0.1	0.03	0.2	0.075	0.4	0.199	0.6	0.368	0.8	0.596	0.9	0.747	1.86
Hungary	0.1	0.039	0.2	0.088	0.4	0.213	0.6	0.379	0.8	0.602	0.9	0.752	1.81
India	0.1	0.035	0.2	0.081	0.4	0.197	0.6	0.347	0.8	0.54	0.9	0.665	2.14
Italy	0.1	0.035	0.2	0.087	0.4	0.227	0.6	0.408	0.8	0.637	0.9	0.782	1.67
Japan	0.1	0.048	0.2	0.106	0.4	0.248	0.6	0.424	0.8	0.644	0.9	0.783	1.61
Korean Rep.	0.1	0.029	0.2	0.075	0.4	0.204	0.6	0.378	0.8	0.607	0.9	0.757	1.81
Mexico	0.1	0.014	0.2	0.036	0.4	0.108	0.6	0.226	0.8	0.418	0.9	0.572	3.05
Netherlands	0.1	0.028	0.2	0.073	0.4	0.2	0.6	0.372	0.8	0.6	0.9	0.749	1.85
Nigeria	0.1	0.016	0.2	0.044	0.4	0.126	0.6	0.251	0.8	0.444	0.9	0.592	2.82
Norway	0.1	0.041	0.2	0.1	0.4	0.243	0.6	0.422	0.8	0.646	0.9	0.788	1.61
Poland	0.1	0.03	0.2	0.077	0.4	0.203	0.6	0.37	0.8	0.591	0.9	0.737	1.88
Portugal	0.1	0.031	0.2	0.073	0.4	0.189	0.6	0.348	0.8	0.566	0.9	0.716	2.00
Russian Fed.	0.1	0.017	0.2	0.044	0.4	0.13	0.6	0.263	0.8	0.464	0.9	0.613	2.68
S. Africa	0.1	0.011	0.2	0.029	0.4	0.084	0.6	0.176	0.8	0.353	0.9	0.541	3.56
Slovakia	0.1	0.051	0.2	0.119	0.4	0.277	0.6	0.463	0.8	0.685	0.9	0.818	1.45
Spain	0.1	0.028	0.2	0.075	0.4	0.201	0.6	0.371	0.8	0.598	0.9	0.748	1.85
Sweden	0.1	0.037	0.2	0.096	0.4	0.241	0.6	0.422	0.8	0.654	0.9	0.799	1.59
Switzerland	0.1	0.026	0.2	0.069	0.4	0.196	0.6	0.369	0.8	0.598	0.9	0.748	1.86
USA	0.1	0.015	0.2	0.048	0.4	0.153	0.6	0.313	0.8	0.548	0.9	0.715	2.14
Venezuela	0.1	0.013	0.2	0.037	0.4	0.121	0.6	0.257	0.8	0.469	0.9	0.63	2.64

None of the welfare distributions can be claimed to be exactly of type F_a for specified values of parameter a since the regression objective differs from zero in all cases. The statistics of Lorenz curve fitting is treated, for example, in Schluter and Trede (2000). The average contribution $\bar{\Delta}$ to the error term is $\sum_{i=1}^{6} \Delta_i^2 = 6\bar{\Delta}^2$ which is bounded from below by the best fit case. In that case $0.00206 = \sum_{i=1}^{6} \Delta_i^2 = 6\bar{\Delta}^2$ so that $\bar{\Delta} = \sqrt{.00206/6} = 0.0185$. The average error margin thus is slightly greater than 1.8% of the welfare. Though other types of regression curves might yield better goodness of fit (in some reasonable measure),

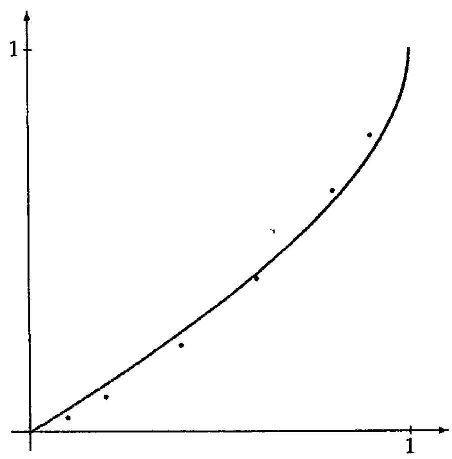

Figure 3. Sketch of the regression curve $F_{1.7}(x)$ for Germany. The two rightmost sample points lie above the curve while all other sample points lie below.

the present curves are continued to be investigated since they solve the appealing differential equation from Section 2.1.

Slovakia and Brazil represent the extreme cases in terms of the equity parameter. However, for the time being, one would not call Slovakia a successful economy. Therefore, among the successful economies, Austria with $\varepsilon \approx 0.65$ seems to have the highest equity on the globe. European nations, Canada, Japan, and Korea typically have smaller equity parameters than the other nations. Slovakia as well as (interestingly) the Czech Republic on one side and USA on the other side represent the extreme cases in terms of goodness of fit among all nations investigated. In tendency, the regression error increases with the equity parameter since a linear regression of the foregoing errors over the equity parameter has positive slope.

Equity parameters, corresponding equity Gini indices, and empirical Gini indices are given in Table 4. The equity Gini indices are computed by $a/(a+1) - 0.5$, cf. Section 2.1, and empirical Gini indices as given by Worldbank (2001) are divided by 2 for comparability. An observation, without theoretical underpinning, is that all empirical Gini indices in Table 4 exceed the corresponding equity Gini indices.

To, again, illustrate the whole approach which allows to relate individuals' economical situations with statistical parameters of nations, a 40%-individual is considered. This individual is characterized by 40% of the population having less and 60% having more income. For equity parameter 0.5525 (Canada), the income of the 40%-individual approximately is 55% of the average income of the 60% population "above" him. Such a statement cannot be inferred from a mere statistical coefficient like the Gini index.

However, the ranking of all nations under investigation by increasing empirical Gini indices differs only slightly from the ranking by decreasing equity parameters. The cause of possible rank differences is that different Lorenz curves may have the same Gini index but different equity parameters as obtained from regression. For example, the Lorenz curves $F(x) = 1 - \sqrt{1-x}$ and $F(x) = x^2$ both have the same Gini index $1/6 = \int_0^1 x - F(x)\,dx$. The first Lorenz curve is exactly $F_a(x)$ for $a = 2$. Thus, a regression by curves of type

Table 3. Equity parameters, their inverses and corresponding regression errors.

Nation	ε	a	err
Austria	0.6493	1.54	.00378
Brazil	0.2778	3.60	.00902
Canada	0.5525	1.81	.00677
China	0.4464	2.24	.00842
Czech Rep.	0.6173	1.62	.00206
Denmark	0.6289	1.59	.00423
Finland	0.6135	1.63	.00301
France	0.5376	1.86	.00643
Germany	0.5882	1.70	.00441
Gr. Britain	0.5025	1.99	.00693
Greece	0.5376	1.86	.00586
Hungary	0.5225	1.81	.00371
India	0.4673	2.14	.00897
Italy	0.5988	1.67	.00516
Japan	0.6211	1.61	.00209
Korean Rep.	0.5525	1.81	.00627
Mexico	0.3279	3.05	.00668
Netherlands	0.5405	1.85	.00618
Nigeria	0.3546	2.82	.00496
Norway	0.6211	1.61	.00328
Poland	0.5319	1.88	.00449
Portugal	0.5000	2.00	.00439
Russian Fed.	0.3731	2.68	.00584
S. Africa	0.2809	3.56	.01070
Slovakia	0.6897	1.45	.00206
Spain	0.5405	1.85	.00584
Sweden	0.6289	1.59	.00481
Switzerland	0.5376	1.86	.00693
USA	0.4673	2.14	.01197
Venezuela	0.3788	2.64	.00942

$F_a(x)$ reproduces the value $a = 2$ within the computational precision of the regression. Regression of the second curve over the support set used earlier leads to the data from Table 5.

The best fit value for the equity parameter thus is $\varepsilon = 1/a = 0.5649$ which differs from the value $\varepsilon = 0.5$ for the first curve. The values $a = 1.77$ and $a = 2$ differ by more than 0.01 which is the precision of the regression.

Table 4. Equity parameters, their inverses and Gini indices.

Nation	ε	a	Equity Gini index	Empirical Gini index
Austria	0.6493	1.54	.1063	.116
Brazil	0.2778	3.60	.2826	.300
Canada	0.5525	1.81	.1441	.158
China	0.4464	2.24	.1913	.202
Czech Rep.	0.6173	1.62	.1183	.127
Denmark	0.6289	1.59	.1139	.124
Finland	0.6135	1.63	.1198	.128
France	0.5376	1.86	.1503	.164
Germany	0.5882	1.70	.1296	.150
Gr. Britain	0.5025	1.99	.1656	.181
Greece	0.5376	1.86	.1503	.164
Hungary	0.5225	1.81	.1441	.154
India	0.4673	2.14	.1815	.189
Italy	0.5988	1.67	.1255	.137
Japan	0.6211	1.61	.1168	.125
Korean Rep.	0.5525	1.81	.1441	.158
Mexico	0.3279	3.05	.2531	.269
Netherlands	0.5405	1.85	.1491	.163
Nigeria	0.3546	2.82	.2382	.253
Norway	0.6211	1.61	.1168	.129
Poland	0.5319	1.88	.1528	.165
Portugal	0.5000	2.00	.1667	.178
Russian Fed.	0.3731	2.68	.2282	.244
S. Africa	0.2809	3.56	.2807	.297
Slovakia	0.6897	1.45	.0918	.098
Spain	0.5405	1.85	.1491	.163
Sweden	0.6289	1.59	.1139	.125
Switzerland	0.5376	1.86	.1503	.166
USA	0.4673	2.14	.1815	.204
Venezuela	0.3788	2.64	.2253	.244

Table 5. Support data for different Lorenz curves with same Gini index and different regression results.

Nation	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	x_6	y_6	a
$F(x) = x^2$	0.1	0.01	0.2	0.04	0.4	0.16	0.6	0.36	0.8	0.64	0.9	0.81	1.77

4. Merging populations

Two economies with parameters a_1, B_1, M_1 and a_2, B_2, M_2, respectively, are to be merged. The values a_1^{-1}, a_2^{-1} indicate the equity parameters; B_1, B_2 denote total income in absolute terms; and M_1, M_2 denote population size. B_1/M_1 and B_2/M_2 denote absolute per capita income. The issue is to compute the joint Lorenz curve. As exact computations are infeasible for practically all non-trivial constellations, approximations and bounds are given.

4.1. Exact solution

The x quantile, $0 < x < 1$, of the poorest of the joint population typically consists of i_0 and j_0 individuals of the poorest of each population. For sufficiently small values of x the poorest stem from only one population; this case will become clear from the following. The mixing case is made up of the smallest individual welfare values

$$\frac{B_1}{M_1} f_{a_1}(0), \quad \frac{B_1}{M_1} f_{a_1}\left(\frac{1}{M_1}\right), \ldots, \frac{B_1}{M_1} f_{a_1}\left(\frac{i_0 - 1}{M_1}\right)$$

and

$$\frac{B_2}{M_2} f_{a_2}(0), \quad \frac{B_2}{M_2} f_{a_2}\left(\frac{1}{M_2}\right), \ldots, \frac{B_2}{M_2} f_{a_2}\left(\frac{j_0 - 1}{M_2}\right),$$

where

$$\frac{i_0 + j_0}{M_1 + M_2} \approx x.$$

The discrete proportions are smoothed meaning that

$$x_1 + x_2 = x \quad \text{with } x_1 \approx \frac{i_0}{M_1 + M_2} \quad \text{and} \quad x_2 \approx \frac{j_0}{M_1 + M_2},$$

with quantiles x_1, x_2 of the merged population being computed by exact (continuous!) equality of welfare

$$\frac{B_1}{M_1} f_{a_1}\left(\frac{i_0}{M_1}\right) \approx \frac{B_1}{M_1} f_{a_1}\left(x_1 \frac{M_1 + M_2}{M_1}\right) = \frac{B_2}{M_2} f_{a_2}\left(x_2 \frac{M_1 + M_2}{M_2}\right)$$
$$\approx \frac{B_2}{M_2} f_{a_2}\left(\frac{j_0}{M_2}\right).$$

This decomposition is the key to analyse the merger of two populations. The decomposition can always be facilitated:

$$\frac{B_1}{M_1} f_{a_1}\left(x_1 \frac{M_1 + M_2}{M_1}\right) = \frac{B_2}{M_2} f_{a_2}\left(x_2 \frac{M_1 + M_2}{M_2}\right) = \frac{B_2}{M_2} f_{a_2}\left((x - x_1) \frac{M_1 + M_2}{M_2}\right)$$
$$\Leftrightarrow \frac{B_1}{M_1 a_1}\left(1 - x_1 \frac{M_1 + M_2}{M_1}\right)^{\frac{1}{a_1} - 1} = \frac{B_2}{M_2 a_2}\left(1 - (x - x_1) \frac{M_1 + M_2}{M_2}\right)^{\frac{1}{a_2} - 1}.$$

The equation has a unique solution in the typical case (otherwise it is unsolvable; criterion for separating cases, c.f. above), since the densities f_{a_1}, f_{a_2} are strictly increasing and, thus, the last two functions (in x_1) are strictly monotone in opposite directions. Solving for x_1 requires a numerical root finder, such as regula falsi. For arbitrary x, the values x_1, x_2 are assumed to be known.

The absolute welfare of the x quantile of the poorest of the joint population equals

$$\begin{aligned}
&\frac{B_1}{M_1} f_{a_1}(0) + \frac{B_1}{M_1} f_{a_1}\left(\frac{1}{M_1}\right) + \cdots + \frac{B_1}{M_1} f_{a_1}\left(\frac{i_0 - 1}{M_1}\right) \\
&\quad + \frac{B_2}{M_2} f_{a_2}(0) + \frac{B_2}{M_2} f_{a_2}\left(\frac{1}{M_2}\right) + \cdots + \frac{B_2}{M_2} f_{a_2}\left(\frac{i_0 - 1}{M_2}\right) \\
&= \sum_{i=1}^{i_0} \frac{1}{M_1} B_1 f_{a_1}\left(\frac{i-1}{M_1}\right) + \sum_{j=1}^{j_0} \frac{1}{M_2} B_2 f_{a_2}\left(\frac{j-1}{M_2}\right) \\
&\approx \int_0^{\frac{i_0}{M_1}} B_1 f_{a_1}(u)\,du + \int_0^{\frac{j_0}{M_2}} B_2 f_{a_2}(u)\,du \\
&= \int_0^{\frac{i_0}{M_1+M_2}\frac{M_1+M_2}{M_1}} B_1 f_{a_1}(u)\,du + \int_0^{\frac{j_0}{M_1+M_2}\frac{M_1+M_2}{M_2}} B_2 f_{a_2}(u)\,du \\
&\approx B_1 F_{a_1}\left(x_1 \frac{M_1 + M_2}{M_1}\right) + B_2 F_{a_2}\left(x_2 \frac{M_1 + M_2}{M_2}\right).
\end{aligned}$$

In summary, this results in relative welfare, normalized to $0 - 1$, of the x quantile of the poorest of the joint population

$$F(x) = \frac{B_1}{B_1 + B_2} F_{a_1}\left(x_1 \frac{M_1 + M_2}{M_1}\right) + \frac{B_2}{B_1 + B_2} F_{a_2}\left(x_2 \frac{M_1 + M_2}{M_2}\right),$$

if x is uniquely decomposable into $x = x_1 + x_2$ with $\frac{B_1}{M_1} f_{a_1}(x_1 \frac{M_1+M_2}{M_1}) = \frac{B_2}{M_2} f_{a_2}(x_2 \frac{M_1+M_2}{M_2})$.

The welfare distributions F_a are not closed under merging populations in general. However, the special case $a_1 = a_2 = a$ and $\frac{B_1}{M_1} = \frac{B_2}{M_2}$ results in $x_1 = \frac{M_1}{M_1+M_2} x$, $x_2 = \frac{M_2}{M_1+M_2} x$ and $F(x) = F_a(x)$. Merging identical populations thus preserves the distribution. This is exactly what Dalton's principle of population requires, (cf. Litchfield, 1999).

The welfare distributions F_a are not even approximately closed under merger. This is meant in the following sense. The merger of two distributions, F_{a_1} and F_{a_2}, is approximated by one distribution F_a according to regression. The optimal equity parameter $\varepsilon = a^{-1}$ then need not lie between the equity parameters $\varepsilon_1 = a_1^{-1}$ and $\varepsilon_2 = a_2^{-1}$. The reason is that individual populations may have similar welfare distributions on clearly different absolute levels. Merging the populations may then cause more unevenness, i.e. a smaller equity parameter than contained in any individual distribution; equivalently, the inter-inequity dominates the intra-inequity. While all nations have an equity parameter ≥ 0.25, the equity value for the globe seems to be less than 0.125.

Table 6. Regression data for EU nations, corresponding parameter a and regression error.

Nation	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	x_6	y_6	a
EU	0.1	0.0267	0.2	0.0732	0.4	0.198	0.6	0.3535	0.8	0.618	0.9	0.7527	1.83

Nation	a	err
EU	1.83	.00821

4.2. *Approximate solutions*

The equity parameter of a merger can be approximated by assuming that income is constant over all segments under consideration. These income data are weighted by absolute values such as by individual nation's gross income. The latter must be approximated if not directly available. This allows to merge the welfare segments of the populations by mere sorting. The resulting data can be used as a base for regression. Regression data for 14 nations of the European Union, the resulting parameter a, and the regression error are given in Table 6.

An approximation of the joint Lorenz curve can be obtained by approximating rather than by exactly computing the mixing ratio of the populations. Absolute population sizes are ignored. The weights by which the individual Lorenz curves influence the joint curve are naturally chosen to be

$$\frac{F_{a_1}(x)}{F_{a_1}(x)+F_{a_2}(x)} \quad \text{and} \quad \frac{F_{a_2}(x)}{F_{a_1}(x)+F_{a_2}(x)}$$

respectively. Taking convex combinations results in the simple approximation formula

$$F(x)=\frac{F_{a_1}(x)}{F_{a_1}(x)+F_{a_2}(x)}F_{a_1}(x)+\frac{F_{a_2}(x)}{F_{a_1}(x)+F_{a_2}(x)}F_{a_2}(x)=\frac{F^2_{a_1}(x)+F^2_{a_2}(x)}{F_{a_1}(x)+F_{a_2}(x)}.$$

The approximation is sandwiched by the original Lorenz curves, since $\min\{F_{a_1}(x), F_{a_2}(x)\} \le F(x) \le \max\{F_{a_1}(x), F_{a_2}(x)\}$. This relation need not be true for the merger of two populations of clearly distinct welfare.

For clearly distinct welfare, it is supposed that the richest of the poor population, say the first, is even poorer than the poorest of the rich population. This can be considered as an extreme form of welfare apartheid. Strictly, this assumption contradicts the assumption that both original populations have Lorenz curves of type $F_a(\cdot)$. Their densities being unbounded suggests that welfare has no upper bound, so that some fraction of the poor population has more welfare than some other fraction of the rich population. Making the strict assumption, anyway, induces that the implicit merging step of the exact computation for the joint Lorenz curve becomes obsolete.

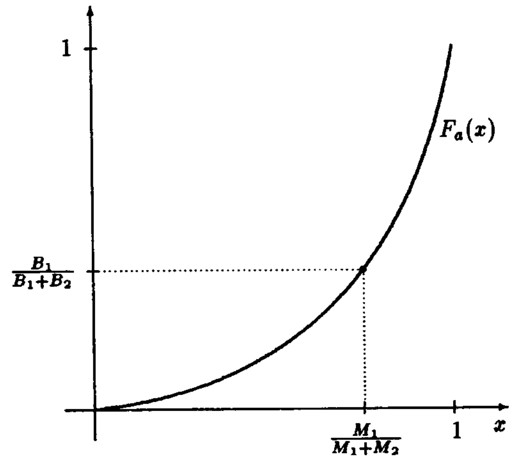

Figure 4. Relative population and their relative welfare determining the joint Lorenz curve and, hence, approximating the equity parameter.

Assuming that the joint Lorenz curve is of type $F_a(\cdot)$ allows one to compute the equity parameter from the single equation

$$F_a\left(\frac{M_1}{M_1+M_2}\right) = \frac{B_1}{B_1+B_2},$$

(cf. figure 4).

This yields an approximation of the equity parameter by

$$1-\left(1-\frac{M_1}{M_1+M_2}\right)^{\frac{1}{a}} = \frac{B_1}{B_1+B_2} \Leftrightarrow a = \frac{\log \frac{M_2}{M_1+M_2}}{\log \frac{B_2}{B_1+B_2}} = \frac{\log(M_1+M_2) - \log M_2}{\log(B_1+B_2) - \log B_2}.$$

The approximation considers the individual populations as so much different that their individual equity parameters (both intra-equities) do not matter.

Sample computations. The poor population is assumed to consist of half of the overall population and to have 20% of the overall welfare. Then, $M_1 = 0.5(M_1 + M_2)$ and $B_1 = 0.2(B_1 + B_2)$ which leads to $F_a(.5) = .2 \Leftrightarrow a = 3.106$. The poor population, consisting of 80% of the overall population and having 20% of the overall welfare, leads to $M_1 = 0.8(M_1 + M_2)$, $B_1 = 0.2(B_1 + B_2)$ and $F_a(.8) = .2 \Leftrightarrow a = 7.211$.

Making some heroic assumptions to fill in missing data in the Worldbank data and making a similar computation as the previous for the EU, indicates that the world equity parameter is indeed $\varepsilon = a^{-1} \leq 1/8$.

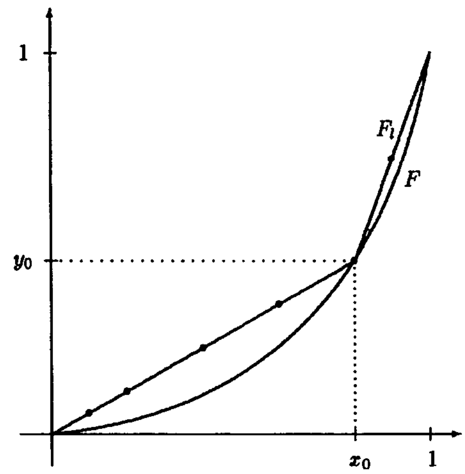

Figure 5. Unknown Lorenz curve F, which is not necessarily of type F_a, and bounding piecewise linear curve F_l with several samples. The samples serve for regression.

4.3. Upper bounds

To avoid overfitting to a single known value of the Lorenz curve, the curve is bounded pointwise from above by the smallest dominating Lorenz curve that is agreeable with the single given value. Whenever the Lorenz curve at $x_0 \in (0, 1)$ attains the value $y_0 \in (0, x_0)$, convexity implies that the curve F_l, consisting of the two straight line segments between $(0, 0)$ and (x_0, y_0), and between (x_0, y_0) and $(1, 1)$, lies pointwise above the unknown curve F (cf. figure 5).

The piecewise linear curve F_l is a Lorenz curve itself, and it can be approximated by a Lorenz curve of type F_a

$$\min_{a>1} \sum_{i=1}^{n} (F_a(x_i) - F_l(x_i))^2.$$

Piecewise linearity of F_l allows one to easily compute the values $F_l(x_i)$ for any support set $\{x_1, \ldots, x_n\}$. Whenever this support set is fixed, the intractable approximation

$$\min_{a>1} \sum_{i=1}^{n} (F_a(x_i) - F(x_i))^2$$

leads to a larger equity parameter than the regression for F_l, since $F(x) \leq F_l(x) \, \forall x \in [0, 1]$, and so the regression curve for F lies below that for F_t. The regression of the piecewise linear Lorenz curve typically does not run through the point (x_0, y_0) but it provides a lower bound for the equity parameter of the intractable regression.

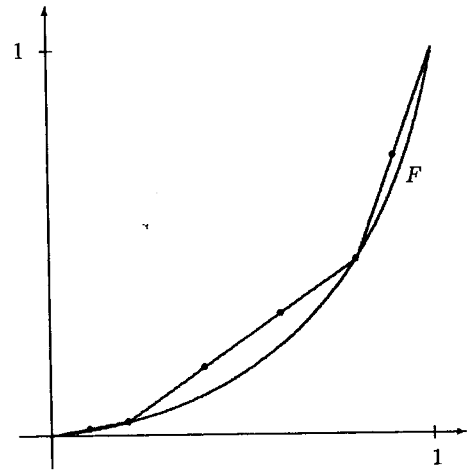

Figure 6. Improvement of the bound sketched in figure 5 leading to an improved bound for the world equity parameter.

Sample computation. The poorest 80% of the world population are assumed to have 20% of the welfare giving $x_0 = .8$ and $y_0 = .2$. These data are not unrealistic for describing today's world (cf. United Nations Population Fund, 2001). The support set is chosen to be the same as for the nation analysis in Section 3.2. A least squares regression then results in an equity parameter of $1 : 3.92$, see below. The world equity parameter can be expected to be smaller than this value.

The bound can be improved once another value of the true Lorenz curve F is given. The lowest 20% of the world population are stated to have 1.3% of the welfare, see United Nations Population Fund (2001). This results in a piecewise linear approximation with three segments as shown in figure 6. The improved upper bound of the world equity parameter is 1:4.05, see Table 7.

The regression errors are about one order of magnitude larger than the errors for the individual nations. The Gini indices of the piecewise linear bounds and the corresponding equity Gini indices are close. The Gini index of the piecewise linear bounds are called empirical Gini indices for consistency. The smaller empirical Gini index coincides with Brazil's which is the largest empirical Gini index of any single nation considered here, see Table 8.

The equity Gini indices for the lower bounds are smaller than the empirical Gini indices in analogy to the nations findings, cf. Section 3.2.

Table 7. Two and three segment linear approximation and best fit values for world data.

Nation	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	x_6	y_6	a
Bound$_1$	0.1	0.025	0.2	0.05	0.4	0.1	0.6	0.15	0.8	0.2	0.9	0.6	3.92
Bound$_2$	0.1	0.0065	0.2	0.013	0.4	0.075	0.6	0.137	0.8	0.2	0.9	0.6	4.05

Table 8. Errors and Gini indices for world data from Table 7.

Nation	ε	a	*err*
Bound_1	0.2551	3.92	.04689
Bound_2	0.2468	4.05	0.5223

Nation	ε	a	Equity Gini index	Empirical Gini index
Bound_1	0.2551	3.92	.2967	.300
Bound_2	0.2468	4.05	.3019	.314

5. Growth and inequity reduction

The most promising potential to reduce inequity is assumed to originate from growth. The view that "economic growth is an indispensable requirement for poverty reduction" is shared by many, see for example UK Secretary of State (2000). Growth itself is not formally modelled here in order to not become susceptible to disputes about growth theories as such. The present view of growth is pragmatic and closer to exogenous than to endogenous concepts as it calls for transfers without a formally rational decision like utility maximization or else by an economical agent.

Quantitative assumptions for the subsequent computations give upper and lower bounds for scenarios. The "upper" bounds basically describe the current state of the world economy while the "lower" bounds describe more or less rigid redistribution understood as investments into empowering. Practical considerations as well as formal economic models confirm that excessive redistribution leads to instability (Dagan and Volij, 2000).

5.1. *Two welfare segments—Why are societies with high inequity relatively poor?*

While complete equity (extreme communism) historically led to poor societies, the same is true for societies with high inequity. Why is this unavoidably the case? Why does this put a limit on the frequently believed pattern that higher inequity amounts to a richer society as a whole? The reason is that some people buy all kinds of (elementary) personal services, if these are cheap, relative to their income. The variety of such services is broad. In certain countries, the personell staff of one household can easily amount to 20 people.

This situation requires people who, with part of their income, pay the income of many other people. Therefore, one has to check to what degree this is possible, depending on the equity parameter. Obviously, for $\varepsilon = .5$, this cannot happen often, as every fully paid person requires already at least 50% of the average income. Within an economy of parameter a, the fraction $1 - x_{high}$ of the population who absorb relative welfare of at least $\frac{n}{a}$, $n = 1, \ldots,$ is called high welfare segment. They are considered to be those who ask for personal services. Its lower boundary is given by

$$f_a(x_{high}) = \frac{n}{a} \Leftrightarrow x_{high} = x_{high;a,n} = 1 - \frac{1}{n^{\frac{a}{a-1}}}.$$

The relative welfare of the high welfare segment is

$$1 - F_a(x_{high}) = 1 - F_a\left(1 - \frac{1}{n^{\frac{a}{a-1}}}\right) = \frac{1}{n^{\frac{1}{a-1}}}.$$

The high welfare segment is supposed to spend $\frac{m}{a}$, $m < n$, on the low welfare segment of the population, namely for requiring elementary personal services, which are paid for as low as possible. To make things easier, we assume the income of the poorest to be completely of this type, so that the welfare of that segment exclusively stems from this spending. The return on this spending is, consequently, called personal services which are provided by the low welfare segment for the high welfare segment. The upper boundary $x_{low} = x_{low;a,n,m}$ of the low welfare segment is the solution of

$$F_a(x_{low}) = \frac{m}{a}(1 - x_{high}).$$

The solution is unique and given by

$$x_{low} = x_{low;a,n,m} = 1 - \left(1 - \frac{m}{a} \cdot \frac{1}{n^{\frac{c}{a-1}}}\right)^a.$$

Parameters n and m are typically chosen to be integers but they may be arbitrary reals greater 1. All parameters are chosen in a reasonable way, meaning that the low welfare segment and the high welfare segment do not overlap, i.e. $x_{low} \leq x_{high}$. Otherwise, the low welfare segment would directly create income from its own consumption. The situation is depicted in figure 7. The sizes of the welfare segments are related by $x_{low} < m(1 - x_{high})$.

Sample computation. The values $a = 3, n = 5$, and $m = 2$ lead to $x_{high} \approx .9105$ and $x_{low} \approx .1694$ meaning that the 9% richest employ the 17% poorest for personal services. In this case, the low welfare segment is nearly twice as large as the high welfare segment. The reason is that the density over the low welfare segment is almost linear with positive,

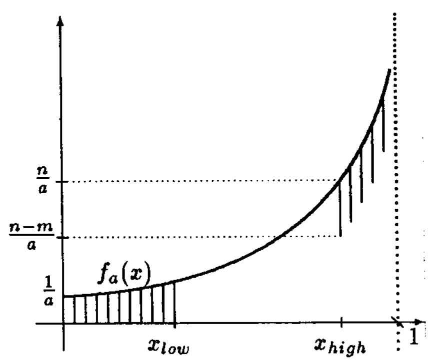

Figure 7. The shaded areas below the density $f_a(x)$ are equally sized and must not overlap. The case of $n = 5$ and $m = 2$ is sketched.

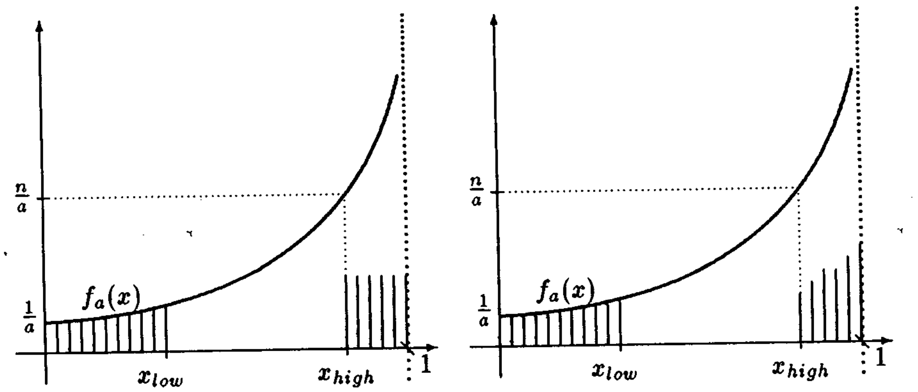

Figure 8. The shaded areas below the density $f_a(x)$ are again non-overlapping and of equal size, here d. The arrangement in the high welfare segment is unspecified, allowing the left and right as well as other cases.

but small, slope. The spending which amounts to each of the shaded areas in figure 7 is $F_3(x_{low}) = 1 - (1 - x_{low})^{\frac{1}{3}} \approx .06$.

5.2. *Two welfare segments: Alternative*

An alternative low welfare segment can be considered from leaving the spending within the high welfare segment unspecified. It is assumed only that a fraction $d, 0 < d < 1$, of the welfare from that segment is spent on the low welfare segment, cf. figure 8. The spending schemes are not identical even for $d = \frac{m}{a}$, since d refers to the welfare of the complete high welfare segment, rather than to the constant difference in $\frac{n-m}{a}$.

The upper boundary of the low welfare segment, now, is the unique solution of

$$F_a(x_{low}) = d \cdot (1 - F_a(x_{high})) = \frac{d}{n^{\frac{1}{a-1}}}$$

which is

$$x_{low} = x_{low;a,n,d} = 1 - \left(1 - \frac{d}{n^{\frac{1}{a-1}}}\right)^a.$$

Sample computation. The values $a = 3, n = 5$, and $d = .1$ lead again to $x_{high} \approx .9105$, but now to $x_{low} \approx .1283$. The spending which amounts to each of the shaded areas in figure 8 now is $F_3(x_{low}) = 1 - (1 - x_{low})^{\frac{1}{3}} \approx .04472$.

Welfare segments and spendings for various parameters are given in Table 9.[2] Inequity is at the highest degrees observed.

5.3. *Two exhaustive segments: The 10 → 4 : 34 formula for sustainable development*

The welfare segments are now assumed to be exhaustive which means that $x_{low} = x_{high}$. Moreover, they are set to $x_{low} = x_{high} = .8$ in accordance with the merger data from

Table 9. Welfare segments x_{low} and spending $F_a(x_{low})$.

	d										
n	.03		.06		.09		.12		.15		x_{high}
	Values for x_{low} and $F_a(x_{low})$ at $a = 3.5$										
5	.0541	.0158	.1060	.0315	.1559	.0472	.2038	.0630	.2497	.0783	.8949
6	..0503	.0146	.0988	.0293	.1456	.0439	.1905	.0586	.2337	.0733	.9186
7	.0474	.0137	.0931	.0275	.1373	.0413	.1799	.0551	.2210	.0688	.9344
8	.0449	.0131	.0885	.0261	.1305	.0391	.1719	.0522	.2104	.0653	.9456
9	.0429	.0125	.0845	.0248	.1248	.0373	.1638	.0498	.2015	.0622	.09538
10	.0411	.0119	.0811	.0238	.1198	.0358	.1574	.0477	.1938	.0596	.9601
	Values for x_{low} and $F_a(x_{low})$ at $a = 3.75$										
5	.0612	.0167	.1197	.0334	.1754	.0501	.2285	.0668	.2790	.0835	.8886
6	.0574	.0156	.1123	.0312	.1849	.0469	.2151	.0625	.2631	.0781	.9131
7	.0543	.0148	.1064	.0296	.1564	.0444	.2044	.0591	.2502	.0739	.0296
8	.0518	.0141	.1016	.0282	.1495	.0423	.1954	.0563	.2395	.0704	.9413
9	.0497	.0135	.0975	.0269	.1435	.0404	.1678	.0539	.2305	.0675	.9500
10	.0478	.0129	.0939	.0297	.1394	.0389	.1813	.0519	.2226	.0649	.9567
	Values for x_{low} and $F_a(x_{low})$ at $a = 4.0$										
5	.0684	.0176	.1331	.0351	.1945	.0526	.2525	.0702	.3073	.0877	.8830
6	.0644	.0165	.1257	.0330	.1837	.0495	.2391	.0660	.2915	.0825	.9082
7	.0613	.0156	.1197	.0313	.1753	.0470	.2282	.0627	.2787	.0784	.9253
8	.0587	.0150	.1147	.0300	.1682	.0450	.2193	.0600	.2679	.0750	.9375
9	.0585	.0144	.1105	.0288	.1621	.0432	.2115	.05678	.2587	.0721	.9466
10	.0545	.0139	.1068	.0278	.1569	.0417	.2048	.0558	.2507	.0696	.9535

Section 4.2. The initial welfare of the low segment is $B_1 = .2B$ so that the initial welfare of the high segment is $B_2 = .8B$.

The overall welfare B is assumed to increase in the long run, say it increases tenfold over 50 or 100 years. This is the idea of the factor 10 concept for global sustainability. The idea is to couple a factor 10 global growth over the time span mentioned with a parallel factor 10 increase in eco-efficiency, so that resource use and environmental degradation are not increased in this process. While markets care for growth and increase in eco-efficiency, the harder part is, politically, a world contract, to keep resource use and pollution within today's limits. It is assumed, that this can either be tried by using force within the frame of an eco-dictatorship of the North against the South, a very dangerous route. Or it can instead be achieved by consensus on a global level; in this case, the North must, however, agree to heavy co-financing to push forward global development to overcome poverty. Certainly, this means asymmetric growth rates between North and South within the factor 10 frame to allow for more catch up and more equity over time.

Table 10. Related growth factors.

b	c
10	10
9	14
8	18
7	22
6	26
5	30
4	34
3	38
2	42
1	46

Consequently, over time, the welfare of the high segment is assumed to increase less than tenfold, say by a factor b with $1 \leq b \leq 10$. Then the welfare of the low segment increases by a factor $c \geq 10$. This unsymmetric situation is in line with so-called pro-poor growth (Kakwani and Pernia, 2000). The individual growth factors are related by $10B = c \cdot B_1 + b \cdot B_2 = c \cdot .2B + b \cdot .8B \Leftrightarrow c = 50 - 4b$. Sample values for the growth factors are given in Table 10.

The long term growth leads to the annual growth rates of Table 11 assuming that growth is constant over time.

The average welfare of an individual from the high segment is $\frac{.8B}{.2M} = 4\frac{B}{M}$, while the average welfare of an individual of the low segment is $\frac{.2B}{.8M} = \frac{1}{4}\frac{B}{M}$. The ratio of these

Table 11. Annual growth rates.

North			South		
Factor b	50 years $\sqrt[50]{b}$	100 years $\sqrt[100]{b}$	Factor c	50 years $\sqrt[50]{c}$	100 years $\sqrt[100]{c}$
10	1.0471	1.0233	10	1.0471	1.0233
9	1.0449	1.0222	14	1.0542	1.0267
8	1.0425	1.0210	18	1.0595	1.0293
7	1.0397	1.0196	22	1.0638	1.0314
6	1.0365	1.0181	26	1.0673	1.0331
5	1.0327	1.0162	30	1.0704	1.0345
4	1.0281	1.0139	34	1.0731	1.0359
3	1.0222	1.0110	38	1.0755	1.0370
2	1.0139	1.0069	42	1.0776	1.0380
1	1.0000	1.0000	46	1.0796	1.0390

Table 12. Growth factors and ratios.

Factor b	Factor c	Initial ratio const	New ratio $b/c \cdot 16$
10	10	16	16.0000
9	14	16	10.2857
8	18	16	7.1111
7	22	16	5.0909
6	26	16	3.6923
5	30	16	2.6667
4	**34**	**16**	**1.8823**
3	38	16	1.2632
2	42	16	0.7619
1	46	16	0.3478

averages is $4\frac{B}{M}/(\frac{1}{4}\frac{B}{M}) = 16$. By growth at different factors per segment, this ratio becomes $b \cdot 4\frac{B}{M}/(c \cdot \frac{1}{4}\frac{B}{M}) = \frac{b}{c} \cdot 16$. Individual growth at different rates may cause segment changes for parts of the overall population. For computational simplification, this is not considered.

Sample values for ratios of average welfare are given in Table 12. Note, that values for the factor b of 2 or lower lead to an inversion of average welfare in the "high" and "low" segments. This is formally admissable for a mere growth estimate but far from being realistic. Moreover, it would be admissable in formal relation to equity only if the roles of the segments were modified, see below. For a number of reasons, $b = 4$ (aside to $b = 5$ or $b = 6$) seems to be a promising scenario. It is discussed in Radermacher (2001, 2002) as the $10 \to 4:34$ formula of global sustainability.

Related equity parameters can be estimated from below by the following computation. Initially, the ratio of the welfare segments is given by

$$\frac{1 - F_a(0.8)}{F_a(0.8)} = \frac{B_2}{B_1} = \frac{0.8B}{0.2B} = 4.$$

By growth, this ratio becomes

$$\frac{1 - F_a(0.8)}{F_a(0.8)} = \frac{b \cdot .8B}{c \cdot .2B} = 4\frac{b}{c}.$$

The latter leads to

$$F_a(.8) = \frac{1}{4\frac{b}{c} + 1} = \frac{.2c}{.8b + .2c} = 1 - .08b$$

which allows a piecewise linear upper bound for the Lorenz curve and subsequent regression to obtain a lower bound for the equity parameter as in Section 4.2. An exact computation of the equity parameter from the value $F_a(.8)$ would lead to overfitting. The bounds for the equity parameter are summarized in Table 13. The function F_a being a Lorenz curve

Table 13. Lower bounds for inequity.

Factor b	Factor c	Lower bound for $c = \varepsilon^{-1}$
10	10	3.92
9	14	3.20
8	18	2.64
7	22	2.20
6	26	1.85
5	30	1.55
4	34	1.30
3	38	1.13
2	42	–
1	46	–

requires $F_a(.8) \leq .8 \Leftrightarrow b \geq 2.5$. Thus, the last two cases cannot be related to equity without reshuffling the order of or within the welfare segments. This would require further assumptions which are not made here.

Details of the regression of the bounding Lorenz curve are provided in Table 14.

Under the previous assumptions the "after growth" situation for $b = 4$ and $c = 34$, which corresponds to the $10 \rightarrow 4 : 34$ scenario, leads to $B_1 = .68 \cdot 10B$ and $B_2 = .32 \cdot 10B$. Furthermore, it is assumed that the low segment population has a population of $M_1 = 7.25$ billion out of 9 billion with individual equity parameter $a_1 = 2\,(\varepsilon_1 = 0.5)$ and the high welfare segment has a population of $M_2 = 1.75$ billion with equity parameter $a_2 = 1.7\,(\varepsilon_2 = 0.5882)$. The equity parameter for the merger can then be approximated (not bounded) by assuming that welfare is locally constant for all individuals in each of the quintiles and in each of the extreme deciles. This allows to compute the merger of all quantiles which provide the base for regression (cf. Table 15).

Table 14. Regression data for bounds of Lorenz curves consisting of two linear segments that meet at $x_5 = 0.8$.

b	c	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	x_6	y_6	a
10	10	0.1	0.025	0.2	0.05	0.4	0.1	0.6	0.15	0.8	0.2	0.9	0.6	3.92
9	14	0.1	0.035	0.2	0.07	0.4	0.14	0.6	0.21	0.8	0.28	0.9	0.64	3.20
8	18	0.1	0.045	0.2	0.09	0.4	0.18	0.6	0.27	0.8	0.36	0.9	0.68	2.64
7	22	0.1	0.055	0.2	0.11	0.4	0.22	0.6	0.33	0.8	0.44	0.9	0.72	2.20
6	26	0.1	0.065	0.2	0.13	0.4	0.26	0.6	0.39	0.8	0.52	0.9	0.76	1.85
5	30	0.1	0.075	0.2	0.15	0.4	0.30	0.6	0.45	0.8	0.6	0.9	0.8	1.55
4	34	0.1	0.085	0.2	0.17	0.4	0.34	0.6	0.51	0.8	0.68	0.9	0.84	1.30
3	38	0.1	0.095	0.2	0.19	0.4	0.38	0.6	0.57	0.8	0.72	0.9	0.88	1.13

Table 15. "After growth" regression data and result.

Nation	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	x_6	y_6	a
merger	0.1	0.043	0.2	0.0913	0.4	0.1996	0.6	0.3406	0.8	0.544	0.9	0.7028	2.05

This means, that the $10 \to 4 : 34$ approach will, in the end, lead to a global equity situation ($a = 2.05$ or $\varepsilon = 0.4878$) which is compatible with the European approach to equity today. Actually that might provide the basis to implement a world democracy.

5.4. *Further analysis*

How is the middle class affected by changes of the equity parameter? One approach to middle class will formally be defined such that it contains the 1-crossings of the densities which are the values $x_1 = x_1(a)$ with $f_a(x_1) = 1$. The 1-crossings describe the population segment $[0, x_1(a)]$ whose individual welfare is at most equal to the overall average. The 1-crossings are explicitly given by $x_1(a) = 1 - \frac{1}{a^{a/(a-1)}}$ for $a > 1$. Values hereof are listed in Table 16

Obviously, opting for societies with inequity means that most people (at least 2/3, typically many more) earn less than average, in order for some to earn more than average. Even harder,

Table 16. 1-crossings as function of inequity.

Inverse equity parameter $a = \varepsilon^{-1}$	1-crossing of density $f_a 1 - \frac{1}{a^{\frac{a}{a-1}}}$
1.0	0...1
1.2	0.6651
1.4	0.6919
1.6	0.7144
1.8	0.7335
2.0	0.7500
2.2	0.7644
2.4	0.7770
2.6	0.7883
2.8	0.7984
3.0	0.8075
3.2	0.8158
3.4	0.8233
3.6	0.8302
3.8	0.8366
4.0	0.8425

Table 17. Cumulative welfare of different sections of middle class.

Inverse equity parameter a	Middle class $F_a(0.9) - F_a(0.6)$	Upper middle class $F_a(0.9) - F_a(0.8)$	Lower middle class $F_a(0.8) - F_a(0.6)$
1.0	0.3000	0.1000	0.2000
1.2	0.3192	0.1147	0.2045
1.4	0.3266	0.1237	0.2029
1.6	0.3268	0.1286	0.1982
1.8	0.3238	0.1307	0.1931
2.0	0.3162	0.1310	0.1852
2.2	0.3082	0.1300	0.1782
2.4	0.2995	0.1283	0.1712
2.6	0.2905	0.1260	0.1645
2.8	0.2815	0.1234	0.1581
3.0	0.2726	0.1206	0.1520
3.2	0.2640	0.1178	0.1462
3.4	0.2545	0.1149	0.1396
3.6	0.2478	0.1120	0.1358
3.8	0.2402	0.1092	0.1310
4.0	0.2329	0.1064	0.1265
Theoretical maximum at	$a = 1.504$	$a = 1.935$	$a = 1.231$

though the 1-crossings are increasing in the inverse equity parameter, the cumulative welfare over all poorer is decreasing in this parameter, i.e. the function $F_a(x_1(a))$ is decreasing in a; even the function $x_1(a) \cdot F_a(x_1(a))$ is decreasing in a. That means, with higher inequity ever more people fall below average and the overall income of this increasing number of people also falls down from e.g. 54.3% for $a = 1.6$ to 50.0% for $a = 2.0$. Noteworthy, the 1-crossings indicate the locations of the maximum differences between the main diagonal and the Lorenz curves, i.e. $\arg\max_{x \in [0,1]} x - F_a(x) = x_1(a)$.

Another formulation of middle class is to regard it as a fixed segment in the distribution pattern of a society. In the following, we consider the middle class to be the population interval [0.6, 0.9] which contains the 1-crossings for all equity parameters up to value 4. The middle class is split into upper middle class [0.8, 0.9] and lower middle class [0.6, 0.8]. Their cumulative welfare as a function of the equity parameter is stated in Table 17.

Though middle class here consists of only 30% of the population around the 1-crossing, it contains heterogeneity. The cumulative welfare of the upper middle class is quite flat so individuals from this class may have no clear preference for a specific equity parameter. The situation is different for overall and lower middle class favouring relatively high equity parameters. This reflects empirical experience that (lower) middle class is hurt if inequity increases too much ("loss of middle ground"). In particular, passing $a = 2(\varepsilon = 0.5)$ the upper middle class loses concerning its relative share, and for values $a > 4(\varepsilon < 0.25)$ will even fall below its 10% share of the extreme equity situation.

The cumulative welfare over any interval $[x_1, x_2]$ with $0 < x_1 < x_2 < 1$ is theoretically maximized for some finite value of the equity parameter which is computable by

$$\arg\max_{a \geq 1} F_a(x_2) - F_a(x_1) \max\left\{ \frac{\ln \frac{1-x_1}{1-x_2}}{\ln \frac{\ln(1-x_2)}{\ln(1-x_1)}}, 1 \right\}.$$

Any increase of the lower interval bound and any increase of the upper interval bound lead to an increase of the maximizing value of the equity parameter.

A more formal consideration of middle class can be based on the lower limit of the 1-crossings $\lim_{a \to 1} +x_1(a) = 1 - 1/e \approx 0.6321$. Density values $f_a(x)$ are decreasing in parameter a for all $x \in (0, 1 - \frac{1}{e})$. All individuals in this segment lose relative income when the inverse equity parameter increases from value 1 onwards. For all $x \in (0, 1 - \frac{1}{e}, 1)$ the density values increase before they decrease for a sufficiently large inverse equity parameter depending on x. All individuals in this segment gain relative income before they start losing when the inverse equity parameter increases from value 1 onwards.

Considering the two welfare segments $(0, 1 - \frac{1}{e})$ and $(1 - \frac{1}{e}, 1)$ raises the fairness issue of their cumulative welfare shares being equal. The inverse equity parameter leading to equality is given by

$$F_a\left(1 - \frac{1}{e}\right) = \frac{1}{2} \Leftrightarrow a = \frac{1}{\ln 2} \approx 1.443.$$

5.5. *Direct growth-equity relations*

A functional dependency between growth rate and equity parameter is assumed for developed nations in the subsequent scenario. The intention is to get an idea of an acceptable level of inequity or to understand why societies use to stay within their equity pattern if there is no other pressure for change.

We now restrict ourselves to the developed world and to its characteristic range of equity parameters which reaches from 1.5^{-1} to 2.2^{-1}. Within this range, it is tempting to assume though debateable that growth increases with inequity. A moderate degree of inequity allows many individuals to perceive that effort will be rewarded. This fosters growth. An increase beyond a certain, yet unknown level excludes such large population segments from growth benefits (with implications for general human resource development) that overall growth loss takes place. As a consequence, growth rates decrease with inequity once a certain level is exceeded. This was discussed in some detail in Sections 5.2 and 5.3, showing that societies with high inequity will necessarily be relatively poor. To make a connection between growth and inequity is not easy. Practically, the US success may also have to do with its superiority in world political affairs, with making use of population growth via newcomers, cannibalization of global resources etc. European people, even over a long run, do not have the feeling that the US people overall live better.

Lack of empirical data forces one to assume plausible relationships instead of, say, regressions of growth over inequity. Results are to be used with due caution.

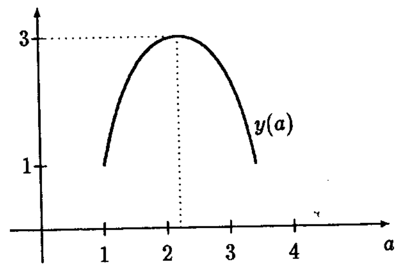

Figure 9. One possible growth-equity relationship.

One relation between growth rate and equity parameter is expressed by the parabola

$$y = y(a) = -1.388a^2 + 6.11a - 3.722$$

for $a \in [1, 3.4]$, cf. figure 9. The extreme values of $a = 1$ and $a = 3.4$ are assumed to lead to an annual growth of 1%, while the maximum growth of 3% per annum is attained for $a = 2.2$. Sample values of annual growth rates are stated in Table 18.

One might ask why people should want to shift from, say $a = 1.6$ to $a = 1.8$. The promise is a little higher growth rate of 0.28%. While this is attractive from the very beginning for those few percent better off (more share plus more growth), for most people it means a double loss on short notice (relative loss and absolute loss) and in the long run a relative loss. How long does it take to have a gain in absolute terms for the poorer part of society? As Table 19 shows, it will be 43 years for the sample transition from $a = 1.6$ to $a = 1.8$.

Note first, that distributional decrease according to increasing inequity will in the end always be over-compensated by compound interest of growth. The time to break even in a comparison of the poorest under two equity parameters a_1, a_2 with $a_2 > a_1$ is given by

$$\left(1 + \frac{y(a_1)}{100}\right)^t f_{a_1}(0) = \left(1 + \frac{y(a_2)}{100}\right)^t f_{a_2}(0) \Leftrightarrow t = \frac{\log a_2 - \log a_1}{\log\left(1 + \frac{y(a_2)}{100}\right) - \log\left(1 + \frac{y(a_1)}{100}\right)}.$$

Table 18. Growth rates as function of equity.

Equity a	Annual growth rate $y(a)$ %
1.0 or 3.4	1.00
1.2 or 3.2	1.61
1.4 or 3.0	2.11
1.6 or 2.8	2.50
1.8 or 2.6	2.78
2.0 or 2.4	2.95
2.2	3.00

Table 19. Break even times as function of change in equity.

Time to break even (in years)						
a_2	a_1					
	1.0	1.2	1.4	1.6	1.8	2.0
1.2	30.278	–	–	–	–	–
1.4	30.783	31.404	–	–	–	–
1.6	31.881	32.988	35.028	–	–	–
1.8	33.645	35.415	38.427	43.176	–	–
2.0	36.247	38.989	43.535	50.938	63.752	–
2.2	40.210	44.611	52.082	65.442	93.985	196.291

Equality of weighted densities at zero suffices to compute the break even point. The reason is that two weighted densities $c_1 f_{a_1}(x)$ and $c_2 f_{a_2}(x)$ with $a_2 > a_1$ and $c_1 f_{a_1}(0) = c_2 f_{a_2}(0)$ do not intersect for any argument strictly between zero and one.

Sample values for the time to break even are given subsequently. After that time, a larger inverse equity parameter (more inequity) is favourable due to the accumulated growth. Each individual is then better off in absolute terms under more inequity than under less. Thus, in the long run, an increase in inequity within the stated range complies with Rawls' theory of justice (Rawls, 1970), but only in absolute terms not in relative, and only after quite some time.

The cumulative welfare of any population segment excluding the wealthiest considered over a fixed time span has the value $(1 + y(a)/100)^t F_a(x)$. This also applies to more general, unimodal functions $y : [1, \infty) \to \mathbb{R}_>$ with maximum value $y_0 = y(a_0)$ attained at $a_0 > 1$. The previous parabola is covered as special case with $a_0 = 2.2$ and $y_0 = 3$. The maximization problem

$$\max_{a \geq 1} \left(1 + \frac{y(a)}{100}\right)^t F_a(x)$$

for fixed $0 < x < 1$ and fixed $t \geq 1$ has optimal value $a < a_0$. This optimum is a solution of the mixed polynomial-exponential equation

$$\frac{t}{100} y'(a) F_a(x) = \left(1 + \frac{y(a)}{100}\right) F_a'(x) \cdot \frac{1}{a}(x - 1) \log(1 - x).$$

The solution satisfies $a < a_0$, since $y'(a) \leq 0$ for $a \geq a_0$ and since the right hand side of the equation is strictly positive for $0 < x < 1$. It appears to be impossible to solve the equation in closed form. Anyway, it is formally established that sacrificing a fraction of the overall maximum growth rate is "optimal" in a reasonable sense.

The segment of the population who benefit from an increase in inequity from a_1 to a_2 with $a_1 < a_2 \leq a_0$ is of the form $(e(t), 1)$. The lower interval limit can be

Table 20. Beneficiaries of an increase in inequity.

t	$e_{a_1,a_2}(t)$
1	.8361
2	.8306
3	.8249
4	.8190
5	.8129
10	.7793
20	.6928
30	.5724
40	.4048
50	.1715

computed by

$$e(t) = e_{a_1,a_2}(t) = \max\left\{0, 1 - \left(\frac{a_1}{a_2} \cdot \left(\frac{1 + \frac{y(a_2)}{100}}{1 + \frac{y(a_1)}{100}}\right)^t\right)^{\frac{a_2 \cdot a_1}{a_2 - a_1}}\right\}.$$

Sample computations for $a_1 = 1.7$, $a_2 = 2.0$ with values $y(1.7) = 2.65$ and $y(2.0) = 2.95$ taken from the foregoing parabola are given in Table 20.

All in all, it becomes understandable that developed societies are very reluctant to opt for change to higher inequity, e.g. to US patterns, in a democratic process, whatever their actual inequity level is. While the loss is immediate for most people, the potential gains lie somewhere in the future, often far in the future. Only if pressures, e.g. from the world market, are felt, then change happens, as now in Europe, but very reluctantly and with a lot of political resistance.

6. Conclusion and outlook

To conclude with, the paper gives a new mathematical tool to understand the issue of equity better. With this, we now have one more instrument to analyse the world-wide political situation. This is used in developing a "landing platform" 2020 or 2030 e.g. in policy consultations. It is interesting to understand that the present WTO frame seems to offer countries in development only one option to increase wealth, namely to force on its people on the one hand inequity levels in order to be of interest for international investments. China is doing this with the power of its Communist party and accepts an inequity factor below the US level. The Chinese know that this creates a very dangerous internal situation, but it is the only chance for fast development, though at a high risk. But there is little reasonable alternative under present WTO conditions, as reasonable global confinancing is missing.

Similarly, the globalization process seems to split the European societies and the Japanese society, so there is increasing inequity in the North and in the South, combined with the global destruction of environment to achieve a limited degree of convergence towards more equity on the global scale via the WTO process.

This can obviously not work, this is a very critical situation which appears to be avoidable, i.e. a reasonable global equity could be achieved without creating more inequity in the North and in the South and without destroying the global environment. The instruments given in this paper help us to understand those developments better and to come better to terms with what has to be done for a more peaceful development towards sustainability. This leads into the direction of a double-factor-10 approach, combined with an asymmetric growth potential of factor 4 for the North and 34 for the South over the next 50–100 years. In the end, this would lead to an eco-social market situation with a European type of equity, offering opportunities for world democracy, world civil rights and a sustainable future.

Certainly, more work has to be done concerning the concepts touched in this paper. Work on different kinds of empirical data, taking better into account social systems and adapted internal prices in developing countries, looking more into the historical development of the equity factors in different countries, better analyzing the European situation, etc. Such work is under way in form of further studies by the authors and their co-workers and in the framework of EU projects such as TERRA 2000.

Finally, the new instrument can help to better understand empirical findings concerning equity and the middle class, consequences of Rawls' Theory of Justice for equity issues, and how and when societies might opt voluntarily for more inequity than they presently have.

Notes

1. Computations for all subsequent regression tables and all error tables were provided by Dirk Bank (FAW Ulm) using MATLAB.
2. Computations provided by Thomas Schauer (FAW Ulm).

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