Proportionality-induced distribution laws

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Abstract

Lorenz curves for income distributions are developed in a systematic way by relating individual incomes to averages of certain income groups. Each such distribution law yields an ordinary differential equation. Many of the differential equations allow explicit solutions leading to parametric kinds of Lorenz curves.

Key words: income distribution, Lorenz curve, differential equations.

1 Introduction

Most explicit Lorenz curves that are used throughout economics and other areas are adapted from probability theory or statistics. But Lorenz curves often describe income inequality and sometimes wealth inequality. It is thus a straightforward issue to investigate if Lorenz curves can be derived from income-based assumptions or even income-based "first principles". Proportionality of each income to various income averages will be shown to lead to various parametric Lorenz curves of which many appear to be new.

The most prominent case is that of the Pareto distribution. Each income being proportional to the average of all larger incomes specifies the distribution type completely. The proportional relationship between individual incomes and averages of all larger incomes is occasionally denoted as Wijk's law and that law is framed in a differential equation for the income distribution function. Alternatively, the law can be derived from a differential equation for Lorenz curves and this differential equation admits many variations. All variations relate individual incomes to averages of larger or smaller incomes, in some cases to both. These relationships will be understood as proportionality laws.

Solutions of ordinary differential equations for proportionality laws can be given in closed form for many instances but they refuse solution attacks in few others. Closed form solutions amount to parametric types of Lorenz curves. These can be used for general discussions and fitting to empirical data in a straightforward manner.

The remainder of the paper is organized as follows. The proportionality law of the Pareto distribution is discussed in some detail in section 2. It is generalized in section 3 leading to a system of differential equations. Their solutions are given in section 4 together with a brief empirical investigation and quite a few variations. These include differential equations for which explicit forms of the proportionality law are unknown. Section 5 gives a result overview and section 6 draws a conclusion.

2 Approach

2.1 The Pareto law

Within an income distribution to be constructed, every income can be assumed to be proportional to the average of all larger incomes. The underlying proportionality factor is denoted as equity parameter ε and it ranges throughout $(0,1]$. The following calculus is suited for normative and descriptive views alike and it is governed by the proportionality assumption.

As individual incomes are proportional to the change in values of the Lorenz curve, the desired proportionality law can be stated in terms of a Lorenz curve and its derivative where the derivative exists. More precisely, differentiable Lorenz curves with the desired proportionality law satisfy the linear inhomogeneous differential equation

$$L'(u) = \varepsilon \cdot \frac{1-L(u)}{1-u}.$$

The solution is, see [KPR]

$$L(u) = 1-(1-u)^{\varepsilon}.$$

This Lorenz curve belongs to a Pareto distribution with single parameter ε. In general, the Pareto distribution is two-parametric with shape parameter $\alpha > 0$ and range parameter $x_m > 0$. Density and distribution function are

$$\begin{aligned} f(x) &= \begin{cases} \alpha \cdot (1/x_m)^{-\alpha} \cdot x^{-(\alpha+1)}, & \text{if } x \geq x_m \\ 0, & \text{if } x < x_m \end{cases} \\ F(x) &= \begin{cases} 1-(\frac{x}{x_m})^{-\alpha}, & \text{if } x \geq x_m \\ 0, & \text{if } x < x_m \end{cases} . \end{aligned}$$

The Pareto distribution is a so-called heavy-tail distribution so that expectations, variances and higher moments may not be finite. The expectation is only finite for shape parameters $\alpha > 1$ and, if so, has the value $x_m\alpha/(\alpha-1)$. The variance is only finite for larger shape parameters $\alpha > 2$ and, if so, has the value $x_m^2\alpha/((\alpha-1)^2(\alpha-2))$. Interestingly, the Lorenz curve of the two-parametric Pareto distribution is independent from the range parameter and only depends on the shape parameter $L(u) = 1-(1-u)^{-1/\alpha+1}$ with $\alpha > 1$. All two-parametric Pareto distributions with same ratio of minimum income over mean income have the same Lorenz curve. The ratio equals $\frac{x_m}{x_m\,\alpha}(\alpha-1) = -1/\alpha+1$.

We will always couple the two parameters of the Pareto distribution in the following way: the range parameter is set equal to the equity parameter which indicates the minimum income; $x_m = \varepsilon$. The shape parameter is then adjusted so that the expected income becomes one; $\frac{\varepsilon\cdot\alpha}{\alpha-1} = 1 \iff \alpha = 1/(1-\varepsilon) \iff \epsilon = -1/\alpha+1$. A one-parametric Pareto distribution is thus singled out from the two-parametric Pareto distributions by the ratio of minimum and mean income being equal to the equity parameter. Though the one-parametric Pareto distributions form a proper subset of the two-parametric Pareto distributions with finite mean, they have identical sets of Lorenz curves. The restriction to the one-parametric situation which is implied by the differential equation for the Lorenz curve, is thus not essential.

For equity parameters strictly smaller than one, the income density and the distribution function are, respectively

$$\begin{aligned} f(x) &= \begin{cases} \frac{1}{1-\varepsilon} \cdot \varepsilon^{1/(1-\varepsilon)} \cdot x^{1/(\varepsilon-1)-1}, & \text{if } x \geq \varepsilon \\ 0, & \text{if } x < \varepsilon \end{cases} \\ F(x) &= \begin{cases} 1-(\frac{x}{\varepsilon})^{1/(\varepsilon-1)}, & \text{if } x \geq \varepsilon \\ 0, & \text{if } x < \varepsilon \end{cases} . \end{aligned}$$

This one-parametric Pareto distribution has finite mean value one for all equity parameters $0 < \varepsilon < 1$ and it has finite variance $(1-\varepsilon)^2/(2\varepsilon-1))$ only for equity parameters $0.5 < \varepsilon < 1$. The case $\varepsilon = 1$ is given separately by a single point distribution. A Pareto density and a distribution function are sketched in figure 1.

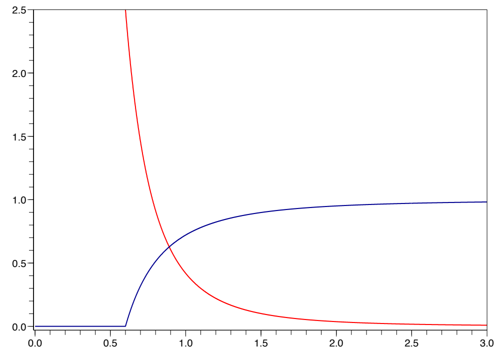

Figure 1: Density (decreasing) and distribution function (increasing) of the Pareto distribution with equity parameter $\varepsilon = 0.6$.

2.2 Related work

The two-parametric Pareto distribution with shape parameter greater one is the only so that each income is proportional to the average of all larger incomes. This so-called Wijk's law has been verified by the differential equation $x \cdot F'(x) + \alpha\, F(x) = \alpha$ with $F(x_m) = 0$ having only one solution, namely the Pareto distribution function [C, p. 155].

Wijk's proportionality law extends a concept of relative poverty. An individual can be consider as being poor if his income falls short of a certain fraction of the average of all incomes. This amounts to Wijk's law for the lowest income. Though this notion of relative poverty has been considered, among others, in official statistics for some time, it has been replaced by a median concept. The German federal government in its first report on poverty and wealth used the four combinations of mean value and median and the 50% and 60% fractions as poverty lines. Later, that report became harmonized with the EU standard so that the 60% fraction of the median was used exclusively [Bu2].

Though also based on differential equations, the present proportionality approach differs from the Burr system which is based on differential equations for distribution functions [Ri]. Some distributions of this system like the Burr (type 12) distribution, also known as Singh-Maddala distribution have been studied in the context of income distributions [McD]. A similar system is the Pearson system which is defined by the densities satisfying certain first order differential equations [Ord]. This system, also, is derived from the statistical perspective rather than from an income perspective.

An overview on parametric classes of Lorenz curves that are not necessarily related to differential equations is given in [Sar].

3 Proportionality laws

3.1 Geometrical interpretation

The differential equation has a geometrical interpretation in terms of tangent slopes of a Lorenz curve, see figure 2. The tangent slope is bounded by two secant slopes as

$$\frac{L(u)}{u} \le L'(u) \le \frac{1 - L(u)}{1 - u}.$$

Intending the upper inequality to become an equation motivates to decrease the upper bound. This is obtained by introducing a positive factor $\varepsilon \le 1$ which results in the foregoing differential equation.

Similarly, intending the lower inequality to become an equation motivates to increase the lower bound which is obtained by introducing a factor $M \geq 1$ which results in the differential equation

$$\frac{L(u)}{u} \cdot M = L'(u).$$

This linear homogenous differential equation has the polynomial solution $L(u) = u^M$. Without knowing this solution and without knowing the Pareto curve being a solution of the other differential equation, the solutions can be verified to be Lorenz curves just from a certain value property.

Lemma 1 *Every solution with $L(u) \leq 1$ of the upper differential equation is increasing and convex and every solution with $L(u) \geq 0$ of the lower differential equation is increasing and convex.*

Proof. The function $\varphi(u) = \frac{1-L(u)}{1-u}$ for $u \in [0, 1)$ is non-negative. It is also increasing which is equivalent to

$$\begin{aligned} \varphi'(u) &= \frac{-L'(u) \cdot (1-u) + 1 - L(u)}{(1-u)^2} \geq 0 \\ \iff \frac{1-L(u)}{1-u} &\geq L'(u). \end{aligned}$$

A solution of the upper differential equation with values not exceeding one satisfies $L'(u) = \varepsilon \cdot \frac{1-L(u)}{1-u} \leq \frac{1-L(u)}{1-u}$. Thus, $L'(u)$ is non-negative and increasing which implies that $L(u)$ is increasing and convex. The argument for the lower differential equation works similarly. ◇

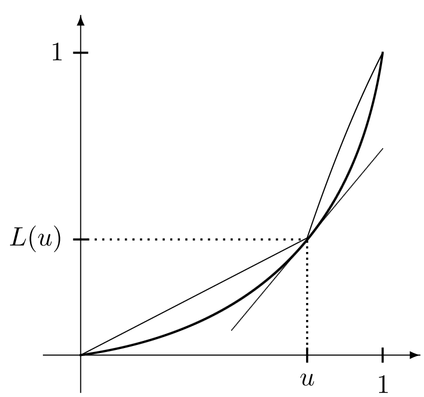

Figure 2: The tangent slope of any Lorenz curve at any interior point is sandwiched by secant slopes as indicated.

In principle, the tangent inequalities could be forced to become equations by addition rather than multiplication. This would lead to the differential equations $\frac{L(u)}{u} + A = L'(u)$ with $A \geq 0$ and $L'(u) = \frac{1-L(u)}{1-u} - a$ with $0 \leq a < 1$. However, these do not result in Lorenz curves with continuos derivatives except for the trivial case $a = A = 0$. This can be seen as follows for Lorenz curves which are continuously differentiable at zero: $L'(0) = \lim_{u \to 0} L'(u) = \lim_{u \to 0} \frac{L(u)}{u} + A = \lim_{u \to 0} \frac{L(u) - L(0)}{u - 0} + A = L'(0) + A$. Thus, these differential equations are not pursued any further.

3.2 The main proportionality laws

The equations $\frac{L(u)}{u} \cdot M = L'(u) = \varepsilon \cdot \frac{1-L(u)}{1-u}$ with $M \geq 1 \geq \varepsilon > 0$ have no common solution except the Egalitarian Lorenz curve $L(u) = u$. Yet we note that the given equations are three and that the polynomial curve and the Pareto curve result from breaking the symmetry to the left and to the right of the three equations, respectively. Crossing out the differential in the middle also results in an equation for Lorenz curves, namely in the functional equation

$$\frac{L(u)}{u} \cdot M = \varepsilon \cdot \frac{1-L(u)}{1-u}.$$

Few simple algebraic manipulations show that the solution is the one-parametric function $L(u) = \frac{u}{M/\varepsilon - (M/\varepsilon - 1)\,u} = \frac{u}{\bar{M} - (\bar{M}-1)\,u}$ with $\bar{M} = M/\varepsilon \geq 1$. It is verified by insertion that this function, which was not derived from differentials, satisfies even two homogenous differential equations

$$(\frac{L(u)}{u})^2 \cdot \bar{M} = L'(u) = \frac{1}{\bar{M}} \cdot (\frac{1-L(u)}{1-u})^2.$$

An income distribution with this Lorenz curve asserts that each income is proportional to the squared average of all larger incomes and, simultaneously, of all smaller incomes! Also, each income equals the average of all larger incomes with proportionality 'factor' being the average of all smaller incomes and vice versa. This can be seen from the Lorenz curve also satisfying the symmetric differential equation of the Bernoulli type

$$L'(u) = \frac{L(u)}{u} \cdot \frac{1-L(u)}{1-u}.$$

Generalizations of the differential equations with squared average incomes can be obtained by raising the average incomes to even higher powers. All in all, this results in the sets of lower and upper differential equations

$$(\frac{L(u)}{u})^n \cdot M = L'(u) \text{ and } L'(u) = m \cdot (\frac{1-L(u)}{1-u})^n$$

for $n = 1, 2, 3, \ldots$ and $M \geq 1 \geq m > 0$. All these differential equations are solvable by Lorenz curves in closed form.

4 Solutions

4.1 Closed form solutions

The lower differential equations have the solutions

$$L_l(u) = \begin{cases} u^M, & \text{if } n = 1 \\ \frac{u}{M-(M-1)\,u}, & \text{if } n = 2 \\ \frac{u}{\sqrt[n-1]{M-(M-1)\,u^{n-1}}}, & \text{if } n = 3, 4, \ldots \end{cases}$$

These solutions can be found with the assistance of the on-line symbolic ODE solver of the Wolfram alpha [Wα]. The solution for $n = 1$ is a simple polygon and the solution for $n = 2$ belongs to the class of hyperbolic Lorenz curves [Ar]. The upper differential equations have the solutions

$$L_u(u) = \begin{cases} 1 - (1-u)^m, & \text{if } n = 1 \\ 1 - \frac{1-u}{m+(1-m)\,(1-u)}, & \text{if } n = 2 \\ 1 - \sqrt[n-1]{1/m} \cdot \frac{1-u}{\sqrt[n-1]{1+(1/m-1)\,(1-u)^{n-1}}}, & \text{if } n = 3, 4, \ldots \end{cases}$$

Though solutions of the upper differential equations can be given 'directly', the solution formulas are quite intricate and their length increases in the power n. The stated expressions were obtained by the function substitution $K(u) = 1 - L(1-u)$. Then $L(u) = 1 - K(1-u)$ and $L'(u) = K'(1-u)$. This allows to rewrite the differential equations as

$$K'(1-u) = L'(u) = m \cdot (\frac{1-L(u)}{1-u})^n = m \cdot (\frac{K(1-u)}{1-u})^n.$$

The variable substitution $w = 1 - u$ results in the lower differential equations except that the proportionality parameter does not exceed one

$$K'(w) = m \cdot (\frac{K(w)}{w})^n.$$

Using again the symbolic ODE solver of the Wolfram alpha and backsubstitution lead to the upper solution functions as stated. Note that the intermediate function $K(u)$ need not and will not be a Lorenz curve. The case $n = 2$ results from the formulas for $n \geq 3$ for both lower and upper Lorenz curves. Yet, stating this case separately better illustrates the function type. Interestingly, for $n = 2$ and $M = 1/m$ lower and upper Lorenz curves are identical since they satisfy the lower as well as the upper differential equation, see above.

Lower and upper Lorenz curves are reflections of each other. The reflection of a Lorenz curve is defined by $L^{ref}(u) = 1 - L^{-1}(1-u)$; graphically, the Lorenz curve is reflected along the minor (downward slanted) diagonal of the unit square. Verifying the following result is omitted since it is tedious but elementary.

Lemma 2 *Let L_l and L_u be lower and upper solutions of the main proportionality laws for the same exponent n and proportionality factors $M = 1/m$. Then $L_l^{ref}(u) = 1 - L_u^{-1}(1-u)$ and $L_u^{ref}(u) = 1 - L_l^{-1}(1-u)$.*

The proof is elementary and tedious and, hence, omitted. Reflected sample curves are shown in figures 3 and curves for one exponent only are shown in figure 4. In the purely algebraic approach, considering powers of average incomes does not lead to other types of Lorenz curves than just considering power one. This means that the functional equation $(\frac{L(u)}{u})^n = \mu \cdot (\frac{1-L(u)}{1-u})^n$ with $\mu \geq 1$ has no more solutions for $n \geq 2$ than for $n = 1$; only the parameter μ has to be replaced by $\sqrt[n]{\mu}$.

4.2 Empirics

Parametric Lorenz curves can be fitted to finite collections of support points by least squares minimization. For an upper Lorenz curve with given power n and given support point collection $(u_i, y_i)_{i \in C}$ this amounts to solving the problem

$$\min_{m \in (0,1]} \sum_{i \in C} \Big(L_u(u_i) - y_i\Big)^2.$$

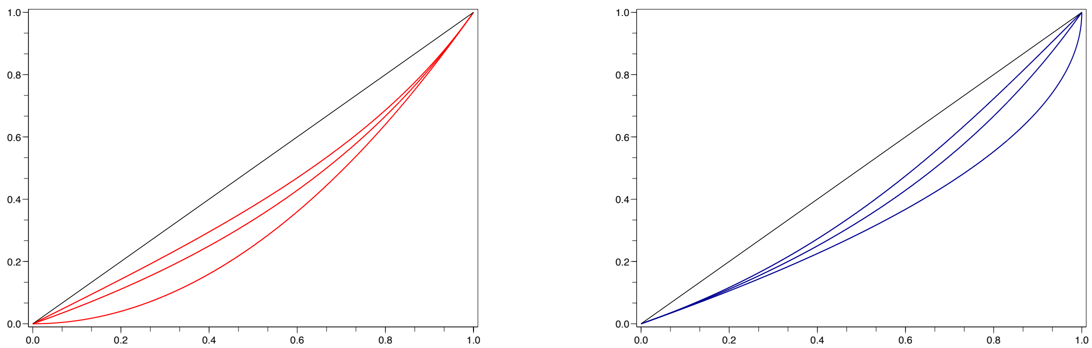

Figure 3: Lower Lorenz curves with $M = 2$ (left) and upper Lorenz curves with $m = 1/2$ (right) for powers $n = 1$ (bottom), $n = 2$ (middle) and $n = 3$ (top) in both cases. The upper Lorenz curve for $n = 1$ is a Pareto curve and the curves for identical exponents are reflections of each other.

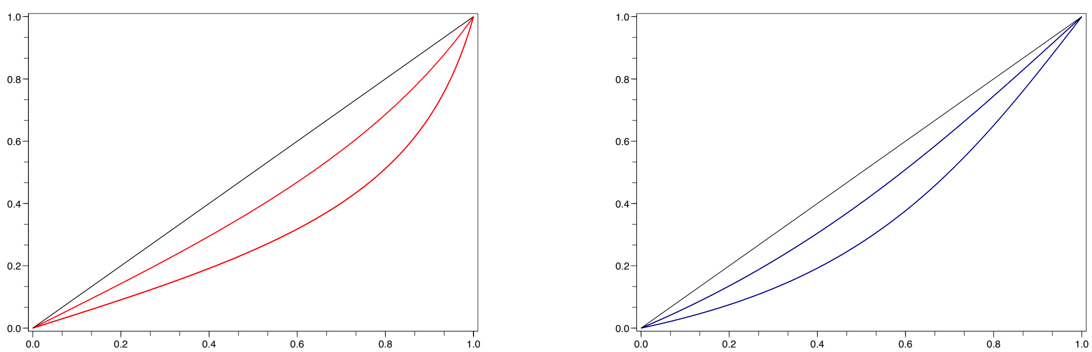

Figure 4: Lower Lorenz curves (left) with $M = 2$ (top) and $M = 5$ (bottom) and upper Lorenz curves (right) with $m = .6$ (top) and $m = .3$ (bottom). The power is set to $n = 3$ for all four curves.

Best least squares fits for the upper Lorenz curves for indices from one through ten have been computed for German income data, see figure 5. The curve for power two resulted in the best overall fit. The same observation can be made for other nations including the US. Noteworthy, the underlying income data span across the whole income range. The well known high fitting quality of the Pareto distribution for top levels of real income data [DS, p. 170] is outweighed by poor fitting elsewhere.

4.3 Further proportionality laws

4.3.1 Fractured exponents

The integer exponents of the income averages can be generalized to genuine rational and genuine real numbers. However, solutions are then intricate to obtain except in special cases. One such case is the lower fractured differential equation $L'(u) = M \cdot (L(u)/u)^{1.5}$ for $M \geq 1$ with Lorenz curve solutions

$$L(u) = \frac{u}{((M-1)\sqrt{u} - M)^2}.$$

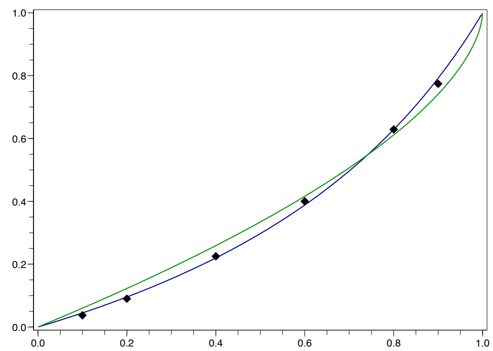

Figure 5: Best fit Pareto curve (mostly top) and best fit upper curve for power $n = 2$ (mostly bottom) for German income data as provided by the Worldbank. The sum of least squares for the upper curve is smaller than for the Pareto curve. Best fit parameters are $m = 0.587$ ('equity parameter') for the Pareto curve and $m = 0.423$ for the upper curve.

Using, again, function substitution and variable substitution, these solutions carry over to solutions of the upper fractured differential equation $L'(u) = m \cdot ((1 - L(u))/(1 - u))^{1.5}$ for $0 < m \leq 1$. Resulting Lorenz curves are

$$L(u) = 1 - \frac{1 - u}{((m-1)\sqrt{1-u} - m)^2}.$$

4.3.2 Proportionality functions

All differential equations of the equity calculus, which were considered so far, have constant coefficients. This means that their proportionality factors are constants. These can be relaxed to functions and differential equation solutions can be given in closed form for certain proportionality functions. Only few explicit examples are given. The lower differential equations $L'(u) = (1 + u^n) \cdot \frac{L(u)}{u}$ with $n = 1, 2, \ldots$ have the solutions

$$L(u) = u \cdot e^{\frac{u^n}{n} - \frac{1}{n}}$$

and the upper differential equations $L'(u) = u^n \cdot \frac{1-L(u)}{1-u}$ with $n = 1, 2, \ldots$ have the solutions

$$L(u) = 1 - (1 - u) \cdot e^{\frac{u^n}{n} + \frac{u^{n-1}}{n-1} + \ldots + u}.$$

Note that polynomial Lorenz curves were used as proportionality functions for the upper differential equations. Even Pareto Lorenz curves can be inserted. An example is $L'(u) = \left(1 - (1-u)^{0.5}\right) \cdot \frac{1-L(u)}{1-u}$ with solution

$$L(u) = 1 - (1 - u) \cdot e^{2 - 2 \cdot \sqrt{1-u}}.$$

The curve is shown in figure 6. All examples fall into the pattern of generating Lorenz curves from given Lorenz curves which serve as proportionality functions in differential equations.

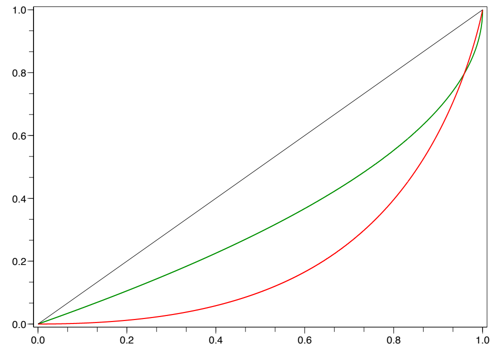

Figure 6: Pareto curve ("upper" curve) with equity parameter 0.5 and Lorenz curve ("lower" curve) using it as proportionality function.

Theorem 1 *Let $\bar{L}(u)$ be some differentiable Lorenz curve. Then*

1. *every solution of $L'(u) = \bar{L}(u) \cdot \frac{1-L(u)}{1-u}$ with $L(0) = 0$, $L(1) = 1$ and $L(u) \leq 1$ and*
2. *every solution of $L'(u) = (1 + \bar{L}(u)) \cdot \frac{L(u)}{u}$ with $L(0) = 0$, $L(1) = 1$ and $L(u) \geq 0$*

is a Lorenz curve.

Proof. Solution functions being increasing and convex is verified similar to lemma 1. ◇

Proportionality functions need not be increasing. For example, the lower differential equation $L'(u) = (2 - 0.1\,u) \cdot L(u)/u$ has the Lorenz curve solution $L(u) = u^2 \cdot exp(0.1 - 0.1\,u)$. Verifying convexity of solution functions can be complicated for decreasing proportionality functions.

4.3.3 Slack functions

The slack between the secant slopes and the tangent slope can be filled-in by certain additive functions, called slack functions though not by constants as discussed towards the end of section 8.1. The lower differential equations with polynomial slack functions $L'(u) = L(u)/u + u^n$ for $n = 1, 2, \ldots$ have the Lorenz curve solutions

$$L(u) = \frac{n-1}{n} \cdot u + \frac{1}{n} \cdot u^{n+1}.$$

The upper differential equations with polynomial slack functions $L'(u) = (1 - L(u))/(1 - u) - (1 - u)^n$ for $n = 1, 2, \ldots$ have the Lorenz curve solutions

$$L(u) = 1 - \frac{n+1}{n} \cdot (1 - u) + \frac{1}{n} \cdot (1 - u)^{n+1}.$$

4.3.4 Averages over other income ranges

Individual incomes can be related to subsets of all larger incomes instead of *all* larger incomes and they can be related to subsets of all smaller incomes instead of *all* smaller incomes. For example, each of these ranges can be cut into half so that an income distribution is characterized by all incomes being proportional to the average over the upper 50% of all smaller incomes. This results in the differential equation

$$L'(u) = M \cdot \frac{L(u) - L(u/2)}{u/2}$$

for $M \geq 1$. The analog holds for all incomes being proportional to the average over the lower 50% of all larger incomes. This results in the differential equation

$$L'(u) = m \cdot \frac{L((1+u)/2) - L(u)}{(1-u)/2}$$

for $m \leq 1$. The geometry for the subset ranges is sketched in figure 7 and the corresponding differential equations are not of any standard type. At best, it is difficult to obtain solutions in closed form. They remain to be found or to be numerically approximated.

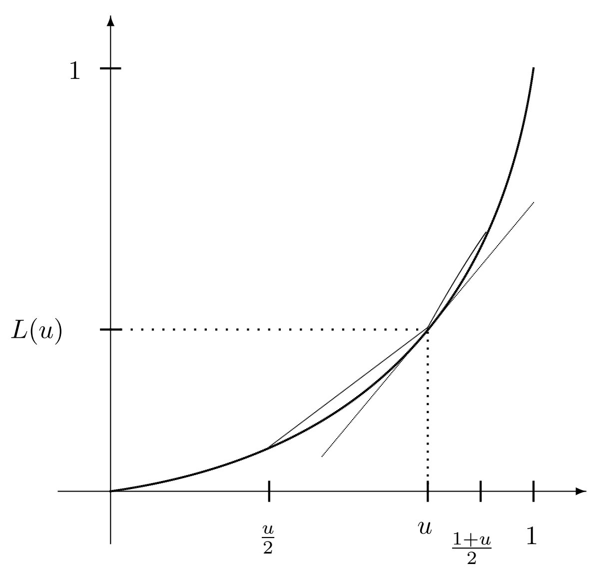

Figure 7: The tangent slope of a Lorenz curve at any interior point is tighter sandwiched by secant slopes than in figure 2.

4.3.5 Limitations

It seems that proportionality factors and proportionality functions are indispensable for obtaining Lorenz curves. Leaving out both requires transformations. The inequalities for the tangent slope of a Lorenz curve can be forced to become equations by transforming the secant slopes to larger or smaller values without using any kind of proportionality.

But solutions may then be trivial as shown for the lower tangent slope: since $0 \leq \frac{L(u)}{u} \leq 1$, any increasing transformation with $T(x) \geq x$ for $x \in [0,1]$ can be used. Feasible examples are $T(x) = \sqrt[n]{x}$ for $n = 2, 3, \ldots$

leading to the Bernoulli differential equations $\sqrt[n]{L(u)/u} = L'(u)$. Similarly, the upper tangent inequality leads to the Bernoulli differential equations $L'(u) = \sqrt[n]{(1-L(u))/(1-u)}$. However, the only Lorenz curve that satisfies any of these differential equations is the Egalitarian curve.

5 System of proportionality laws

As a summary, a system of proportionality laws with their closed form solutions is given in table 1. The system contains all previously considered cases but cannot be claimed to be complete in any reasonable sense.

Proportionality law	Parameter	Solution	Remark
$L'(u) = \varepsilon \cdot \frac{1-L(u)}{1-u}$	$0 < \varepsilon \leq 1$	$L(u) = 1 - (1-u)^{\varepsilon}$	Pareto distribution
$L'(u) = m \cdot (\frac{1-L(u)}{1-u})^2$	$0 < m \leq 1$	$L(u) = 1 - \frac{1-u}{m+(1-m)\,(1-u)}$	$L'(u) = \frac{L(u)}{u} \cdot \frac{1-L(u)}{1-u}$
$L'(u) = m \cdot (\frac{1-L(u)}{1-u})^n$	$0 < m \leq 1$	$L(u) = 1 - \sqrt[n-1]{1/m} \cdot \frac{1-u}{\sqrt[n-1]{1+(1/m-1)\,(1-u)^{n-1}}}$	$n = 3, 4, \ldots$
$L'(u) = m \cdot (\frac{1-L(u)}{1-u})^{1.5}$	$0 < m \leq 1$	$L(u) = 1 - \frac{1-u}{((m-1)\,\sqrt{1-u}-m)^2}$	fractured exponent
$L'(u) = M \cdot \frac{L(u)}{u}$	$M \geq 1$	$L(u) = u^M$	polynomial distr.
$L'(u) = M \cdot (\frac{L(u)}{u})^2$	$M \geq 1$	$L(u) = \frac{u}{M-(M-1)\,u}$	$L'(u) = \frac{L(u)}{u} \cdot \frac{1-L(u)}{1-u}$
$L'(u) = M \cdot (\frac{L(u)}{u})^n$	$M \geq 1$	$L(u) = \frac{u}{\sqrt[n-1]{M-(M-1)\,u^{n-1}}}$	$n = 3, 4, \ldots$
$L'(u) = M \cdot (\frac{L(u)}{u})^{1.5}$	$M \geq 1$	$L(u) = \frac{u}{((M-1)\,\sqrt{u}-M)^2}$	fractured exponent
$L'(u) = (1+u^n) \cdot \frac{L(u)}{u}$	–	$L(u) = u \cdot e^{\frac{u^n}{n} - \frac{1}{n}}$	$n = 1, 2, \ldots$
$L'(u) = u^n \cdot \frac{1-L(u)}{1-u}$	–	$L(u) = 1 - (1-u) \cdot e^{\frac{u^n}{n} + \frac{u^{n-1}}{n-1} + \ldots + u}$	$n = 1, 2, \ldots$
$L'(u) = \big(1 - (1-u)^{0.5}\big) \cdot \frac{1-L(u)}{1-u}$	–	$L(u) = 1 - (1-u) \cdot e^{2-2\cdot\sqrt{1-u}}$	–
$L'(u) = (2 - 0.1\,u) \cdot L(u)/u$	–	$L(u) = u^2 \cdot exp(0.1 - 0.1\,u)$	–
$L'(u) = L(u)/u + u^n$	–	$L(u) = \frac{n-1}{n} \cdot u + \frac{1}{n} \cdot u^{n+1}$	$n = 1, 2, \ldots$
$L'(u) = (1-L(u))/(1-u) - (1-u)^n$	–	$L(u) = 1 - \frac{n+1}{n} \cdot (1-u) + \frac{1}{n} \cdot (1-u)^{n+1}$	$n = 1, 2, \ldots$

Table 1: System of selected, solvable proportionality laws and their explicit solutions.

6 Conclusion

Closed form solutions of differential equation have been shown to result in a variety of parametric Lorenz curves of which quite a few seem t be new. The insight into underlying proportionality laws can help to choose one type of Lorenz curve over another. Also, the "best fit proportionality law" may help in interpretations of curve fitting in empirical studies.

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