Lecture Notes in Economics and Mathematical Systems 679

Thomas Kämpke
Franz Josef Radermacher

Income Modeling and Balancing

A Rigorous Treatment of Distribution Patterns

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In memoriam: Thomas Kämpke

Gone but not forgotten

Dr. habil. Thomas Kämpke

19.07.1957 – 02.01.2015

This book is in memoriam Thomas Kämpke, who died much too early on 2 January 2015 after a long fight with a malicious illness. Thomas Kämpke had a significant share in writing this book, having worked intensively on many intrigue details of the topic. Thomas was an inspiring mathematician, excellent in formal work as well as in applications.

I had the privilege to work with him for almost 30 years, first at the RWTH Aachen, then at the University of Passau, the University of Ulm, the Research Institute for Applied Knowledge Processing (FAW) Ulm and the follow-up institute FAW/n in Ulm. At the University of Ulm, Thomas Kämpke was active for some years as Associate Professor, following his postdoctoral lecture qualification (habilitation) in mathematics. At FAW, he was at first responsible for quite some years for the topic “Computer Science and Environment”, later for the FAW research field “Robotics”. From this last activity resulted his involvement with the successful Ulm start-up InMach, of which he was one of the shareholders and had a senior role concerning the development of new applications and products until his death, similar to his involvement at FAW/n until the end of his active life.

Thomas Kämpke was a reliable partner to all of us. All colleagues involved as well as I myself feel a deep loss. We will miss him and will keep him alive in our memories.

Ulm, Germany Franz Josef Radermacher

All models are wrong
but some are useful
George Box

Foreword

The issue of a balanced income distribution is gaining considerably in importance in the field of politics. Today's income distributions in many nations are no longer acceptable and from a societal point of view counterproductive. Such unbalanced income distributions do not make the life of people better, to the contrary, as I have sought to describe in my scientific work over the years.

Going deeper into the issue, a wish for a better understanding of what is going on concerning income distribution leads to a number of interesting scientific questions in theoretical economics and (partly) applied mathematics. Adequate tools are needed to describe the "nature" of income distributions in an abstract way and to understand how modifications of such distributions correspond to overall economic performance, growth, and relevant societal parameters.

Politically speaking, that has to do with taxation, the financing of the education system, and the operation of the welfare state. But it is not just fiscal measures. Modification of market outcomes is another important field of intervention. One aim is to reduce opportunities for rent seeking by "insiders", among others, by asymmetric access to information, an issue studied by Nobel Laureate Joseph Stiglitz in his work, in which he also broadly tackles the issue of income distributions and their effects on societies.

In seeking to understand these issues, a core result is that derived independently by Serge Kolm and myself linking social values to the ranking of distributions of income (referred to in the literature as the Atkinson theorem). The great majority of people want more balance. Mathematically speaking, that means that we have an overall societal welfare/utility function that is concave in incomes. Then independent of the particular detailed nature of such concave welfare functions, the society will be better off under such a welfare function if (some) income is shifted from richer individuals to poorer ones.

What is more, this can be re-stated in terms of a tool widely used in the empirical study of income distributions: the Lorenz curve. Essentially for each $u \in [0, 1]$, the Lorenz function value $L(u)$ gives the percentage of cumulative income earned by those with $u\%$ lowest incomes. For example, the value for the USA today is $L(0.2) = 0.034$ so that the 20 % lowest incomes received considerably less than

5 % of the total income. Social balance often is described by the share of income of the 80 % lowest incomes. For the USA, this is presently about 50 %. This is low by international standards. It should also be noted that the total share of the 80 % lowest incomes decreased over the last years in many OECD countries, particularly in the USA. Correspondingly, for the 20 % highest incomes, there was a significant increase.

What is the role of the new book by Thomas Kämpke and Franz Josef Radermacher? The two scientists come from the background of rigorous mathematics in the fields of functional analysis and probability theory. As the literature concerning Lorenz curves is very much concentrated on economical applications, the treatment is most often restricted to special income distributions such as discrete ones or those with densities. Kämpke and Radermacher instead deal with all distributions on the (positive) real numbers in the Lebesque sense, i.e. in total generality. The authors are able to derive all classical results, in particular the theorem described above in this broader context. The more abstract level, in a sense, even makes some considerations easier. On this very abstract level, the authors are able to add additional insight into the key results.

The results in this book are primarily of interest for mathematicians, but indirectly also helpful for readers from many fields of science, in particular economics. Certainly, they add an aesthetic dimension to the multitude of work in this field. By the additional insight obtained, they contribute directly to the recent debates that are going on in politics.

From my point of view, with this book the foundation of arguments concerning a proper balance in income distribution in the sense of identifying an “efficient inequality range” has got an additional push from mathematics, which I appreciate very much.

Oxford, UK
2013

Sir Tony Atkinson

Foreword

Having passed through the recent financial crises, people ask how this has been possible. Why could and can some people make a fortune with unfair behavior in markets and why have ordinary people to pay the price? Why become income distributions often more askew in a world of growth and within the governance of a democracy?

Empirically, income distributions in many nations including the USA, but also in Europe, are developing in a way that is unacceptable for the broad public. Such unbalanced income distributions are starting to produce misery instead of progress. I have discussed that over the years in a number of scientific publications. In an even broader sense, the worldwide Occupy movement asks the most crucial question: Why do our democracies not bring about the laws, regulations, and behavior that create more balanced income distributions which would be in the best interest of the great majority of citizens? A lot of insight concerning the issue can, by the way, be found in the recent book "The Price of Inequality" by Joseph Stiglitz, reflecting his experiences as member and chairperson of the Council of Economic Advisers of the Clinton Administration from 1993 to 1997.

The questions raised also aim at science, particularly economics and its role in the processes described. Scientific issues in this context concern the right kind of mathematical tools to describe income distributions and for analyzing their effect on well-being, social balance, economic growth, and sustainability.

I have been working in this field scientifically for decades and have published some classical results as did, e.g., Sir Tony Atkinson from Oxford University. One of his core results is the so-called Atkinson theorem. It applies to situations with too much inequality. The societal welfare/utility function is concave in this situation. Society then will be better off if income is shifted from richer individuals to poorer ones, e.g. via taxation, financing of public goods, and many more interventions by government and people. With view to my own earlier work on the issue, modifying regulation is another important field of intervention. One aim is to reduce opportunities for rent seeking, among others, by asymmetric access to information.

Tools for describing income distributions are connected with the so-called Lorenz curves which are also interesting mathematical objects by themselves. They allow for very deep insights and interesting theorems from the viewpoint of mathematics. This is where the new book by Thomas Kämpke and Franz Josef Radermacher comes into the picture. With their background in the fields of functional analysis and probability theory, they address the issue of Lorenz curves in total generality. On a very abstract level, they are able to derive the classical results in the field for the general case and generalize them considerably concerning content. This, among other issues, concerns not only the above-mentioned so-called Atkinson theorem, but also considerable extensions of classical micro–macro foundations of economics, allowing to replace the representative agent paradigm by arbitrary income distributions in the form of Lorenz curves.

The results in this book are primarily of interest to mathematicians. But with the many applications, this is also true for interested readers from other fields of science. The additional insights obtained will also fit into recent debates on more balanced income distributions and corresponding debates concerning a senseful system of aims for an economy. Hopefully, they may also add new insights into New Economic Theory building.

From my point of view, with this book there are now even more arguments available (this time from mathematics) against too much income inequality. I regard this as very positive. I wish the book many inspired readers and a great impact into the ongoing discussions in society concerning an adequate balance of income distributions.

Karlsruhe, Germany Wolfgang Eichhorn
2014

Preface

This book addresses all scientists and informed lay-persons who are interested in or use the concept of Lorenz curves and who seek a sound understanding of its mathematical basis. In a similar way, the book addresses all scientists and lay-persons interested in income and wealth distributions of nations and their historical development, as discussed in Piketty's recent bestseller "Capital in the Twenty-First Century" (Piketty 2014). Also, it addresses everyone interested in the so-called Atkinson theorem. This important theorem, combined with its dual version, gives deep insight into the issue of income distribution balances in society. This together constitutes Part I of the book (Chaps. 1–6).

Part II of the book (Chaps. 7–9) deals with more special distributions, characterized by special (mathematical) features, such as self-similarity. The insights obtained have consequences for a better understanding of what is required for, e.g., building a majority coalition for societal change in democracies. The matter requires quite some mathematical tools, particularly from probability theory, statistics, theory of real functions, and calculus.

In the field of economy, particularly in the analysis of income distributions, treatment is often restricted to finite discrete distributions and to distributions with Lebesgue density. This treatment is often ad hoc and incomplete. That need not be a problem from the practical sight but is not satisfying from the scientific point of view.

The ambition of the present book is to give a rigorous treatment. Starting from the empirical distribution of a real valued quantity such as income, body mass, and sales volumes (which have nothing to do with probability distributions but may be interpreted as those), we make use of Gastwirth's derivation of Lorenz curves for probability distributions over real numbers or non-negative real numbers, respectively.

The variety of all such distributions is known from Lebesgue's decomposition theorem: every distribution is a mixture of three types, namely a discrete distribution, one with Lebesgue densities, and a singular distribution. The latter distributions have an uncountable support set of Lebesgue measure zero. The real numbers are endowed, as usual, with the Borel sets. For all such distributions, the Lorenz curve can be devised with the same calculus. The main tool is the generalized inverse of a distribution function.

We conclude from our study of related work that many results from the first part of the book can be found in the literature. However, we take a rare, possibly a first look into Lorenz curves of singular measures. Some of the methods and results needed are old and difficult to find. Sometimes, in the literature, details of conditions in formulations do not fit together. We hope to overcome some of these deficiencies.

In Chap. 1, the introduction, concepts from probability are compiled that will be necessary for further analysis. An informal relation is stated between the intuitive and the formal notions of a Lorenz curve.

Key to Lorenz curves is the generalized inverse of a distribution function. This will be covered in Chap. 2 including important properties of generalized inversion, generalized inverse of the generalized inverse, and some convergence results. These include approximations of generally invertible distribution functions by ordinarily invertible distribution functions with such approximations from above being easier than from below.

Lorenz curves together with their derivatives—the Lorenz densities—are introduced in Chap. 3. Well-known characterizations for both are repeated as well as some convergence results are given. Surprisingly, pointwise convergence of distribution functions generally fails to imply convergence of the associated Lorenz curves. But an additional moment condition ensures the desired convergence. In addition, the celebrated Gini index and other indices for income distributions are introduced.

A partial order for Lorenz curves and associated order relations are considered in Chap. 4. A Lorenz curve is understood to be smaller in Lorenz order than another Lorenz curve if the former lies pointwise above the latter. Starting out from majorization of vectors, the convex stochastic order is obtained. Convex stochastic order and Lorenz order are equivalent for distributions with same expectations. The equivalence goes way back in the theory of stochastic orders and it occasionally is attributed to Hardy, Littlewood, and Pólya. An explicit verification of the equivalence is given.

Also, a representation formula for expected utility values is given in terms of expected values and the Lorenz density. Expected utility can thus be thought of consisting of an absolute component—the expectation—and a relative component—the derivative of a Lorenz curve. The representation amounts to a bridge between utility theory and distribution theory. It results, in particular, in a formula for the variance and in a simple model showing that utility maximization leads to underconsumption. The representation formula, also, is helpful when generalizing the so-called standard model of economic theory.

Chapter 5 extends the so-called Pigou–Dalton transfers from their original notion to general probability distributions. Weak convergence of distributions will allow for this extension. Extremely simple swaps thus explain complicated distributional comparisons.

Chapter 6 gives a treatment of the Atkinson theorem. The theorem is stated in several versions that depend on the generality of the underlying distributions. Also, an interpretation in terms of oscillating welfare constellations is given. It is an aim of this book to give Atkinson's theorem in greatest generality and to give proofs.

We include an inverse version of the Atkinson theorem. Only by having both versions, we can fully grasp the empirical phenomena of oscillation of societies around levels of balance. This may be interpreted as corresponding to changes in societal overall utility functions between those of concave and those of convex nature, depending on the state a society feels to be in: either too little balance, which favors concave utility functions and distributes from rich to poor or too much balance, which favors convex utility functions and distributes from poor to rich. Only by combining both insights, it is possible to bring the Atkinson theorem, which is a marvel, into full accordance with empirical findings.

The second part of the book deals with particular Lorenz curves, with ways to derive Lorenz curves (from other Lorenz curves and from certain equations) and with a computational model based on Lorenz curves.

The Pareto distribution is shown in Chap. 7 to be the unique distribution to result from a certain proportionality law and from self-similarity of Lorenz curves. Considering Lorenz curves as distribution functions will allow to study successive Lorenz curve formation whose limiting behavior is related to the Golden section.

Chapter 8 extends the idea of proportionality laws leading to a system of Lorenz curves. This system is based on differential equations of which many can be solved in closed form. Thus, many Lorenz curves can be stated explicitly in parametric form and some of these are believed to be new.

A computational model which relates redistribution to democratic majorities is given in Chap. 9. To obtain a majority, parts of a population may have to be compensated for joining. When incomes are quite balanced, the "cheapest" coalition partners are those with middle incomes, but when incomes already are imbalanced, the "cheapest" coalition partners are those with lowest incomes. Separation of these cases goes along with a bifurcation. This points into the direction of a better understanding of the functioning of democracies in two-class societies as particularly found in many developing countries.

We hope that this work will not only contribute to the continued use of Lorenz curves in empirical studies. It should also be used in political economics to argue for a reasonable order design of markets as advocated for in Eichhorn (1990, 1994), Radermacher (2004a), and Stiglitz (2012) or for eradicating poverty, (Eichhorn and Presse 2012).

Not surprisingly, Lorenz curves and particular details concerning the small high income segment (top-income-segment) play a central role in Piketty's recent bestseller "Capital in the Twenty-First Century" (Piketty 2014). The concentration on the high income segment is also a major issue in Radermacher (2004b) and Radermacher and Beyers (2011). It is also a major argument for using the Pareto distribution in Chap. 8 and in part in Chaps. 7 and 9. This is because the Pareto distribution gives a good fit for the broad picture of any income distribution and, in particular, for the high income segment. This comes at the price of a less good fit in the low income range. We include some hints on the issue dealt with by Piketty in Sect. 7.4.

It is one of the strange phenomena of political communication until recently that instead of discussing the broad picture of income differences between a small rich segment and the rest of the population, there is instead a busy concentration on the details of the low income part. Though the volume distributed there is comparatively small.

Beyond, Lorenz curves should become a tool in economic model building. They provide for one way to overcome the frequent and counterintuitive assumption that all agents are of equal economic strength in case the model aims at utility maximization. That maximization objective is almost prototypic in macro-economic modeling.

The authors thank Sir Tony Atkinson, University of Oxford, and Wolfgang Eichhorn, University of Karlsruhe, for their informed forewords. They also thank Wolfgang B. Jurkat, University of Ulm, for exchange concerning differential equations with regard to Part II of the book. They thank H. Dyckhoff, RWTH Aachen, for motivating this book by insisting on a sound basis for the dissertation work (Herlyn 2012) building on and reflecting some of the material presented here. This material has been collected and developed over the last years. The authors also thank Michael Gerth (FAW/n) for his skillful drawing of many of the figures.

Ulm, Germany Thomas Kämpke
Ulm, Germany Franz Josef Radermacher
2014

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Contents

Acronyms

List of Symbols

$F(x)$	Distribution function
$f(x)$	Lebesgue density (function) or function in general
$F^{-1}(u)$	(Generalized) inverse distribution function
$L(u)$	Lorenz curve
$l(u)$	Lorenz density
EX	Expectation of random variable X
$var\ X$	Variance of random variable X
$L_g(u)$	Generalized Lorenz curve
$L_g'(u)$	Generalized Lorenz density
$\log(x)$	Logarithm function to base e
$\sim$	Proportionality or equality of distributions
$\leq_m$, $\leq_{im}$	Majorization and inverse majorization
$\leq_{gm}$, $\leq_{igm}$	Generalized majorization and inverse generalized majorization
$\leq_{ST}$	Stochastic order
$\leq_{cx}$, $\leq_{cv}$	Convex and concave stochastic order
$\leq_{icx}$, $\leq_{icv}$	Increasing convex and increasing concave stochastic order
$\leq_L$	Lorenz order
$\leq_{PD}$, $\leq_{iPD}$	Pigou–Dalton relation and inverse Pigou–Dalton relation
$1_A(x)$	Indicator function; $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$
$\circ$	Concatenation of functions
$\diamond$	End of a formal argument