Income distribution and majority patterns*

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Abstract

Majority coalitions are formed such that they can redistribute incomes to their favour. When inequality is to be increased in the interest of some, coalition partners must be found by compensation schemes. Compensation minimization is shown to lead to coalition partners being either a connected or a disconnected income group. When inequality reaches certain levels, disconnection becomes unavoidable.

Key words: coalition forming, Lorenz curve, bifurcation, least-cost majority.

1 Introduction

Amartya Sen noted that "... no famine has taken place in the history of the world in a functioning democracy – be it economically rich (as in Western Europe or North America) or relatively poor (as in post independence India, or Botswana or Zimbabwe) ..." [S, p. 16]. This statement sets the scene for the present investigation. Assumptions will be studied that relate to decisions on income distributions with these decisions requiring at least half of all votes.

Majorities in a democracy come in various ways like that of a majority requiring at least fifty per cent or two thirds of all votes. In some voting schemes, which still are denoted as majority voting, a decision may win with even less than fifty per cent of the popular vote. To avoid overly complicated arguments, the fifty per cent mark will always be considered as a necessary majority with the understanding that fifty+ can be obtained with negligible extra effort.

All economical agents have one vote but high income holders are considered to be the drivers in the present investigation. They intend to redistribute income to their favor which amounts to inequality increase. As, typically, less than half the voters benefit from inequality increase, the beneficiaries must seek coalition partners. The main issue, dealt with here, is to identify the coalition partners sought by the winners of inequality increase.

The economical principle to reach a certain goal with least input implies that the potential winners of inequality increase form a coalition with those who are cheapest to compensate for their loss. Formal assumptions will be made to identify these least-cost coalition partners and it will be shown that these can be quite diverse in terms of their current incomes. So diverse, that the best coalition partners in some situations form a connected set while they form a disconnected set in other situations.

Inequality increase will result in bifurcation phenomena between connected and disconnected coalitions. While in some situations only large increases in inequality result in disconnected coalitions, arbitrarily small increases suffice in other situations. The transition from the former to the latter situation can be characterized by a bifurcation phenomenon for one-parametric Lorenz curves. Similar phenomena exist for multi-parametric Lorenz curves.

For the sake of arguments, simplifications must be made. These include the effect of income redistribution on growth being left out. The same applies to possible changes in free trade, free capital flow etc. forced

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upon an economy. All present investigations are conceptual rather than empirical and all models are continuous resulting in some computations being approximate.

The remainder of this work is organized as follows. The formal calculus is stated in section 2 for Pareto-distributions. The main analysis is given for these distributions in section 3 and results are shown to carry over to other distribution types in section 4. All exhibit a certain bifurcation phenomenon in least-cost coalition forming. Section 5 gives a conclusion and an outlook.

2 Model

2.1 General approach

It is assumed here that high income individuals intend to redistribute incomes to their advantage so that high incomes become even higher. This entails – all other things remaining equal – that inequality of the income distribution increases. Under democratic rules, major economical changes such as (considerable) income redistribution require a majority of the people voting for it. As consumers outnumber producers, the consumer view will be predominant here.

Thus, winners of a possible inequality increase either form one group of at least 50% of the voters or they need to form a coalition consisting of at least 50% of all voters. The first situation is unlikely and the second situation requires the original winners to team up with voters who lose in the first place. The latter will demand compensation for their cooperation. Compensation must be paid from gains of inequality increase. This gain will exceed all necessary compensation and is readily available as a premium to be distributed among the winning coalition.

All analyses will be based on Lorenz curves and their derivatives. Because parametric Lorenz curves are more suitable for computations than others, models proceed from one-parametric via two-parametric Lorenz curves to three-parametric which are most general touched upon here.

Income redistribution affects inequality which can be expressed in various ways. Two classical means of expressing inequality are the Gini-index and Lorenz-dominance. One Lorenz curve dominates another if it lies below the other everywhere except at the interval boundaries zero and one where all Lorenz curves have equal values. The dominating Lorenz curve, thus, exhibits greater inequality than the dominated curve. Also, two Lorenz curves that intersect only at the interval boundaries are denoted as comparable. Such concepts will be considered in due course. The smallest inequality is encountered for the identical or Egalitarian distributions where all incomes are equal. Yet, inequality measurement is not the prime issue in analyzing the interests of particular income groups.

2.2 Related work

Two basic modes for income redistribution are common in economic work: (1) transfer and (2) spending on public goods and services [Gro]. The first mode has selected beneficiaries while the second mode, potentially, lets all individuals benefit. The present investigation is closer to the first mode though the perspective here is on structural changes rather than taxes only that favor certain income groups at the expense of the remaining income group. Also, in terms of taxes, redirection is here more important than increase or decrease.

Many investigations of the relation between democracy and income distribution compare nations. A quantitative relation between public sector size, measured by the ratio of tax revenues to GDP, and income inequality, measured by the Gini index, is established by best-fits. This relation is found to be somehow inverse-U shaped for weak democracies and even more vaguely so for fully developed democracies. But findings offer not much explanation and are not ultimately inconclusive [L].

Income distributions resulting from majority voting with implicit coalition forming are given in [Gra]. A notion of stability is established within that approach. Thereby, an income distribution is considered to be stable if a majority may change it but a majority comprising the losers from that change can efficiently advert that change. Only income distributions that are not too uneven are claimed to be stable. While that approach appears to focus on majorities of varying size from a perspective similar to game theory, the present approach considers majorities of fixed size aiming at large cumulative income. The notion of stability will be carried over to the present context.

It has been noted that low income voters may seemingly vote against their economic self-interest but in the interest of high income voters. This is not claimed to be in disregard of economic self-interest but because of inadequate models [V]. Explanations offered include the impact of voters' comparisons within their residence areas rather than the complete election territory. The present approach, also, intends to shed some light on the reason to form coalitions between holders of widely different incomes leaving gaps between.

Self-interest being the dominating factor of preferences over income distributions – not preferences over incomes – is stressed in the experimental study [RW]. However, this study as most work in experimental economics is concerned with artificial incomes rather than real incomes. Deviations from self-interest may become more pronounced for more elaborate settings, see the findings on ultimatum bargaining in [CK] and the work referenced there. For example, anonymity of the distributive process seems to favor self-interest.

Theoretical explanations of self-interest in terms of absolute payoffs being superimposed with relative payoffs are given in [BO]. Only the assessment of both payoffs by a joint preference function, called motivation function, explains revealed distributional decisions. These references as well as the bulk of work in experimental economics seems to center around dictator and ultimatum games. Though the present approach might shed light on coalition forming in such games, that issue is of minor importance here.

The holder of the median income may change by redistribution but the celebrated median voter [Co] is still of some importance in the following. Predominantly, the focus is upon the median voter as on any other coalition member.

The notion of bifurcation, here, refers to parameterized models in general and neither to iterated function systems nor to dynamical systems as frequently considered in economics, see for example [BH]. Bifurcations for parameterized functions are known, for example, from the set of their zeros [HK, p. 84] and their minima [PG].

2.3 Formal approach

The analysis is initialized with the Pareto distribution. This type of distribution is chosen because of its handyness and not because of its empirical quality – which is rather limited. The Pareto distribution has the Lorenz curve $F_\varepsilon(x) = 1 - (1-x)^\varepsilon$ with so-called equity parameter ε ranging in the interval $(0, 1]$. Parameter $\varepsilon = 1$ denotes the identical distribution (all incomes are equal) and decreasing parameters indicate growing unevenness or, synonymously, growing inequality. Growing inequality with respect to the Pareto distribution can equivalently be expressed, among others, by increasing Gini index and decreasing equity parameter. The equity parameter and the Gini index of a Pareto distribution can be computed from each other by the same function: $\varepsilon = (1-G)/(1+G)$ and $G = (1-\varepsilon)/(1+\varepsilon)$.

Whereever a Lorenz curve is differentiable, values of the derivative are proportional to absolute incomes. The Pareto distribution is differentiable except at the right interval bound and the derivative is also denoted as density since $F_\varepsilon(x) = \int_0^x f_\varepsilon(u)\,du$. The density of the Pareto Lorenz curve is convex and it equals $f_\varepsilon(x) = F'_\varepsilon(x) = \varepsilon\,(1-x)^{\varepsilon-1}$ so that $f_\varepsilon(0) = \varepsilon$ and $\lim_{x\to 1-} f_\varepsilon(x) = \infty$. The first equation means that the minimum income stands at the positive level ε or, in absolute terms, the minimum income equals ε times the average income; value one of a density indicates average income. The second equation means that incomes have no upper bound.

Lorenz curves and their densities are related for Pareto distributions by the differential equation $F'(x) = \varepsilon \cdot \frac{1-F(x)}{1-x}$, see [KPR]. Actually, the Pareto distribution is the only income distribution which fulfills this linear differential equation. The interpretation is that each income is proportional to the average of all larger incomes with the equity parameter being the proportionality factor. The latter can be reformulated as all upper truncations of the income distribution being equal which amounts to self-similarity of the Pareto distribution.

The minimum income relative to the average income is given for any distribution, in terms of densities $f(x)$ of Lorenz curves, by $f(0)$ and the median income is given by $f(0.5)$.

3 Redistribution

3.1 Concept

Densities for two different equity parameters are shown in figure 1. Two densities have a unique intersection point that depends on the equity parameters before and after redistribution. The intersection point is also denoted as break even point. All other things remaining equal, incomes above the break even point increase when the equity parameter is decreased. And incomes below the break even point are decreased accordingly. The break point is computable as

$$x_{be} = x_{be}(\epsilon_0, \epsilon_1) = 1 - e^{\frac{\log \epsilon_1 - log \epsilon_0}{\epsilon_0 - \epsilon_1}}$$

and it can be bounded from below as $x_{be} > 1 - e^{-1/\varepsilon_0}$ for all $\varepsilon_1 < \varepsilon_0$. It is stated without argument that the income at a break even point always lies above the average income. Thus, even holders of average incomes lose from inequality increase. Individuals above the break even point will formally be denoted as original winners. The size of their segment tends to zero if the equity parameter after redistribution tends to zero. This is independent from the equity parameter before redistribution; formally $\lim_{\varepsilon_1 \to 0} x_{be}(\epsilon_0, \epsilon_1) = 1$ for all $\varepsilon_0 \leq 1$.

The cumulative gain of all original winners equals the cumulative loss of all losers. Temporarily, income positions are assumed to be unchanged by mere variation of the equity parameter. This means that if one income is smaller than another before redistribution then it will be so after redistribution. This will no longer be true for later compensation and redistribution schemes. Pareto Lorenz curves of a smaller equity parameter always dominate the Pareto Lorenz curve of a larger equity parameter.

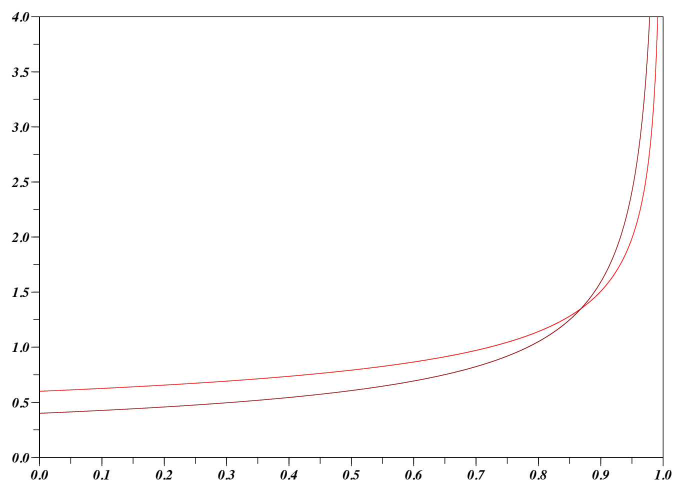

Figure 1: Densities $f_{\varepsilon_0}(x)$ and $f_{\varepsilon_1}(x)$ of the Pareto distribution with initial and terminal equity parameters $\varepsilon_0 = 0.6$ and $\varepsilon_1 = 0.4$. The break even point lies at $x_{be} = 0.8683$ so that the highest 13.17% of all incomes are increased while all incomes below the break even point are decreased.

It will be assumed throughout that redistribution requires a majority of at least 50% of all votes which is simplified to a "majority" of exactly 50%. Results for genuine majorities of strictly more than 50% of all votes can be obtained similar to those given below but result formulations and notation then become more cumbersome without the results becoming essentially different.

All original winners from an inequality increase form a minority only; the break even point lies above 0.5 for all pairs of equity parameters. But the beneficiaries have the common interest of inequality increase so that they can be considered as a core group seeking coalition partners until a majority is reached. Due to the stated simplification, this majority is reached when exactly half of all voters favour the intended inequality increase.

It is straightforward for coalition forming by the original winners to search for those voters who suffer minimum loss by mere inequality increase. These will be cheapest to compensate. The loss of each loser is quantified by the loss function which denotes the income reduction of each loser. The loss function is simply given as the difference of the densities before and after redistribution

$$l(x) = f_{\varepsilon_0}(x) - f_{\varepsilon_1}(x)$$

for $\varepsilon_1 < \varepsilon_0$ and all $x \le x_{be}$. The break even point x_{be}, as specified before, is nothing but the unique root of the loss function. Values of the loss function for the original winners are negative but these values are ignored. Two sample loss functions are sketched in figure 2.

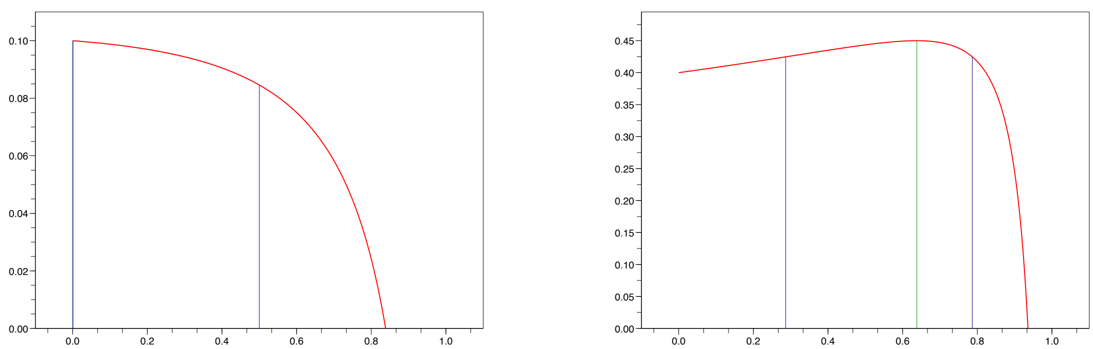

Figure 2: Loss functions for the transitions from the initial equity parameter $\varepsilon_0 = 0.6$ in both cases to $\varepsilon_1 = 0.5$ (left) and $\varepsilon_1 = 0.2$ (right). The maximum loss is suffered at the minimum income for the small decrease (left) and in the upper half for the large decrease (right). The original winners lie in the small intervals between the zero of the loss function and one (not shown). The vertical lines indicate the maximum of the loss function and the 50% interval of the population who suffer the maximum loss.

The more uneven Pareto distribution should be considered as an intermediate distribution only. Redistribution of gains from an inequality increase will lead away from the Pareto distribution even if the original income distribution were exactly of the Pareto type. But, as Pareto distributions typically are approximations of actual income distributions only, both original and redistributed incomes are approximate Pareto distributions.

3.2 Identifying coalition partners with minimum loss

Incomes close to the break even point are cheapest to compensate while incomes close to the maximum of the loss function are most expensive to compensate. The maximum of the loss function lies at the unique position

$$x_m = x_m(\epsilon_0, \epsilon_1) = \max\{0, 1 - e^{\frac{log(\epsilon_1 \cdot (1-\epsilon_1)) - log(\epsilon_0 \cdot (1-\epsilon_0))}{\epsilon_0 - \epsilon_1}}\}.$$

The maximum may lie at zero or somewhere between zero and the break point. Hence, the loss from an increase in inequality can become maximal for the lowest incomes. But when inequality is increased further, lowest incomes are so low that the maximum loss is suffered by middle incomes. Eventually, this drives the point of maximum loss to the upper interval boundary; $\lim_{\varepsilon_1 \to 0} x_m(\epsilon_0, \epsilon_1) = 1$ for all $\varepsilon_0 \leq 1$.

Finding the cheapest coalition partners amounts to fill-in the original winner segment from the break even point downwards while avoiding the maximum region. More precisely, the cheapest coalition partners form a subset C that complements the original winner segment and which is of minimum cumulative loss

$$\min_{C \subseteq [0, x_{be}),\, size(C) = x_{be} - 0.5} \int_C l(x)\, dx.$$

The size of the cheapest coalition is less than half of the loser segment from which it is taken. This follows from $2 \cdot (x_{be} - 0.5) < x_{be}$ if and only if $x_{be} < 1$ with the latter being obviously true. It can further be shown, as a consequence, that the compensation amount for the cheapest coalition is less than half of the gain of the original winners. So, under the present assumptions, the original winners can always afford an inequality increase.

Depending on the two equity parameters before and after redistribution, the cheapest coalition will either be connected or disconnected. It is disconnected if and only if the 50% interval (of the non-coalition partners) which incurs maximum cumulative loss, lies strictly in the interior of the domain. A connected coalition set is shown in figure 2 (left) and a disconnected coalition set is indicated in figure 2 (right).

From the computational viewpoint, the cheapest coalition set is determined by the complementary 50% interval with maximum cumulative loss. The complementary 50% set with maximum cumulative loss actually is an interval around the maximum of the loss function since there is no other maximum. The optimal non-coalition interval $(a, a{+}.5)$ with $0 \leq a < .5$ is hence specified by $\max_{0 \leq a < x_{be} - .5} \int_a^{a+.5} l(x)\, dx$. This interval either satisfies the following optimality condition for its lower bound

$$\frac{d}{d\,a} \int_a^{a+.5} l(x)\, dx = l(a + .5) - l(a) = 0$$

or, if this equation does not have a solution, the optimal non-coalition interval is $(0, .5)$. The cheapest coalition is computed from the non-coalition interval by complementation

$$C = (0, x_{be}) - (a, a + .5) = \begin{cases} [0.5, x_{be}), & \text{if } \ a = 0 \\ (0, a] \cup [a + .5, x_{be}), & \text{if } \ a > 0. \end{cases}$$

The cheapest coalition is now seen to be either a connected or a disconnected set. In both cases, the cheapest coalition comprises those losers who are adjacent to the original winners which means that the cheapest coalition segments reaches up to the break even point. Disconnection occurs always when income inequality is sufficiently increased.

There seems to be no closed form solution of the optimality equation for the cheapest coalition but fast numerical procedures exist. Even successive bisection as sketched next is applicable and fast on this problem.

Coal-comp

1. Input $\varepsilon_0, \varepsilon_1$.
   Initialization $a_l = 0$, $a_u = 0.5$, $a = 0.25$, $\delta = 10^{-12}$ (termination threshold).
2. If $f_{\varepsilon_0}(0) - f_{\varepsilon_1}(0) \geq f_{\varepsilon_0}(.5) - f_{\varepsilon_1}(.5)$ then $a = 0$, else while $a_u - a_l > \delta$ do
   (a) $Diff(a_l) = f_{\varepsilon_0}(a_l + .5) - f_{\varepsilon_0}(a_l) - f_{\varepsilon_1}(a_l + .5) + f_{\varepsilon_1}(a_l)$.
   (b) $Diff(a) = f_{\varepsilon_0}(a + .5) - f_{\varepsilon_0}(a) - f_{\varepsilon_1}(a + .5) + f_{\varepsilon_1}(a)$.
   (c) If $sign(Diff(a_l)) \neq sign(Diff(a))$ then $a_u = a$ else $a_l = a$.

(d) $a = 0.5 \cdot (a_l + a_u)$.

3. Output $C = [0.5, x_{be})$ if $a = 0$, else $C = (0, a] \cup [a + .5, x_{be})$.

The number of iterations made by this coalition finding algorithm is at most $\lceil -\log \delta / \log 2 \rceil$ with $\lceil x \rceil$ denoting the smallest integer greater or equal to x. For the given termination threshold $\delta = 10^{-12}$ the iteration bound equals $\lceil 35.8496 \rceil = 36$.

Disconnection of the cheapest coalition together with the original winners may actually be considered as a coalition of three groups: the original winners, the mild losers and the strong losers. The mild losers, which are formed by the upper interval of the cheapest coalition, have incomes that reach up to those of the original winners. The strong losers, which are formed by the lower interval of the cheapest coalition, comprise the individuals with lowest incomes.

Incomes before redistribution may be more similar between original winners and mild losers than between mild and strong losers. More precisely, the ratio of the average incomes before redistribution of original winners and mild losers may be smaller than the average income ratio of mild towards strong losers. However, depending on the parameters, the opposite is also feasible. Depending on the equity parameters, the average loss of the strong losers can either be less or equal or larger than the average loss of all losers – including the non-coalition segment. Once the cheapest coalition is disconnected, the segment of strong losers increases when inequality increases further.

3.3 Bifurcation

Connectivity of the cheapest coalition can be fully characterized. It is related to the difference between the median income and the minimum income under the Pareto distribution according to the following argument

$$\begin{aligned} \text{Connectivity} &\iff l(0.5) \leq l(0) \\ &\iff f_{\varepsilon_0}(0.5) - f_{\varepsilon_1}(0.5) \leq f_{\varepsilon_0}(0) - f_{\varepsilon_1}(0) \\ &\iff f_{\varepsilon_0}(0.5) - f_{\varepsilon_0}(0) \leq f_{\varepsilon_1}(0.5) - f_{\varepsilon_1}(0). \end{aligned}$$

The differences on both sides of the last inequality are proportional to the difference between median and minimum income according to the parameters ε_0 and ε_1 respectively. When income inequality is increased, connectivity of the cheapest coalition is retained if and only if the difference between median and minimum income in the original distribution has a corresponding difference of at least the same amount under the more uneven distribution.

The difference between these pivotal incomes as function of parameters will be denoted as med-min function − not only for the Pareto distribution − and it is shown in figure 3. A critical point analysis verifies that the function has no inflexion point and no other extremum than its global maximum. If the original equity parameter lies at or below the location of that maximum, then, obviously, no smaller equity parameter entails a difference of at least the original difference value. Thus, there is no connectivity of the cheapest coalition partners in this case. Remarkably, this is even so for arbitrarily small parameter decreases.

The maximizing parameter for the med-min function thus serves as bifurcation parameter

$$\varepsilon_{bif} = argmax_{\varepsilon \in (0,1]} \; f_{\varepsilon}(0.5) - f_{\varepsilon}(0).$$

The so-called partition function serves for separating the two cases of the coalition partners forming a connected and a disconnected set. Formally, the partition function is defined as

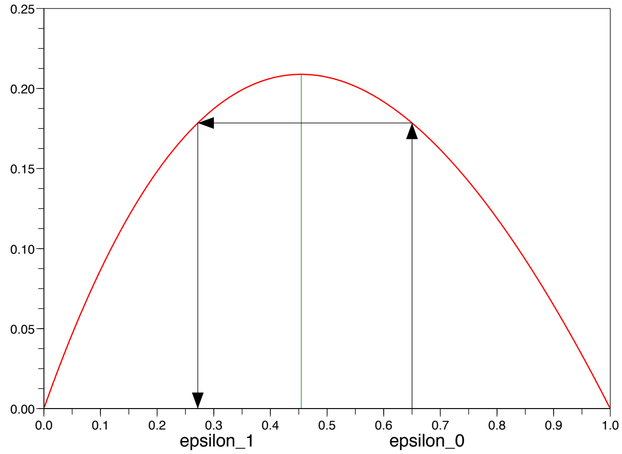

Figure 3: The med-min function $f_\varepsilon(0.5) - f_\varepsilon(0)$ for Pareto distributions becomes maximal at the equity parameter $\varepsilon_{bif} = 0.45433$. The indicated decrease from the sample value $\varepsilon_0 = 0.65$ to the sample value $\varepsilon_1 = 0.2718$ and to each ε' with $\varepsilon_1 < \varepsilon' < \varepsilon_0$ ensures connectivity of the cheapest coalition.

$$pf(\varepsilon_0) = \begin{cases} \varepsilon_0, & \text{if } \varepsilon_0 \leq \varepsilon_{bif} \\ \varepsilon_1, & \text{if } \varepsilon_0 > \varepsilon_{bif,}\varepsilon_1 < \varepsilon_0 \text{ and } f_{\varepsilon_0}(0.5) - f_{\varepsilon_0}(0) = f_{\varepsilon_1}(0.5) - f_{\varepsilon_1}(0) \end{cases}$$

with bifurcation parameter $\varepsilon_{bif} = 0.45433$. A sample value is $pf(0.65) = 0.2718$ as indicated by figure 3. The partition function consists of two segments: a linear increasing segment and a curved decreasing segment. The partition function and its bifurcation are shown in figure 4. The curved segment is not computable in closed form. This circumstance is equivalent to the cheapest coalition not being computable in closed form whenever it is disconnected. Moreover, a thorough analysis reveals that the decreasing segment is curved because the difference function from figure 3 is not symmetric. A concise notion of the partition function is

$$pf(\varepsilon) = \varphi^{-1}(\varphi(\varepsilon))$$

with $\varphi(\varepsilon) = f_\varepsilon(0.5) - f_\varepsilon(0)$ and φ^{-1} being the inverse over the increasing function segment.

The partition function over the curved segment is approximately computable by numerical search procedures like successive bisection similar to the approximation of the bifurcation equity parameter itself. Also, it should be noted that bifurcation, here, is a consequence of an optimality principle.

For clarity, the bifurcation phenomenon is illustrated in figure 5. While a small decrease leads to connectivity of the cheapest coalition from an initial equity parameter above the bifurcation value, the same small decrease causes a split from an initial equity parameter below the bifurcation value. One huge decrease in the equity parameter may result in a connected cheapest coalition while the same decrease, when split into several steps, may entail some cheapest coalitions being disconnected. An example is the single decrease from 0.6 to 0.4 which results in connection. But if the decrease were split into 20 steps of equal size, the last five steps would result in disconnection.

3.4 Situation after compensation but before complete redistribution

Increasing inequality and compensating the cheapest coalition leads away from the Pareto distribution. Also, it may lead to each strong loser having a higher income than any non-coalition member even before

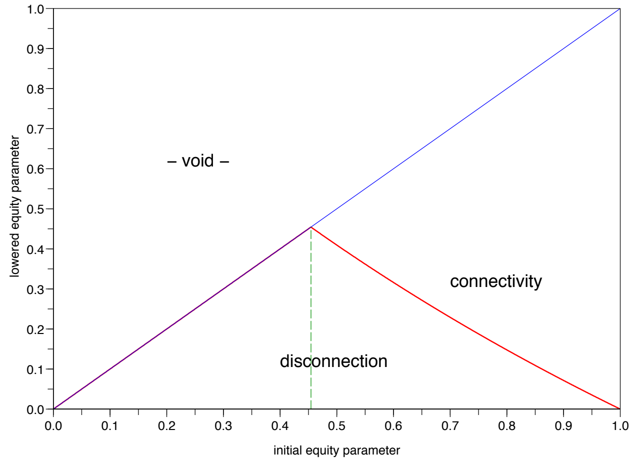

Figure 4: The cheapest coalition partners of the original winners become disconnected when the new equity parameter is driven to or below the partition function. The partition function increases linearly on the main diagonal up to the bifurcation point 0.45433 (vertical dashed line) and decreases strictly convex from there.

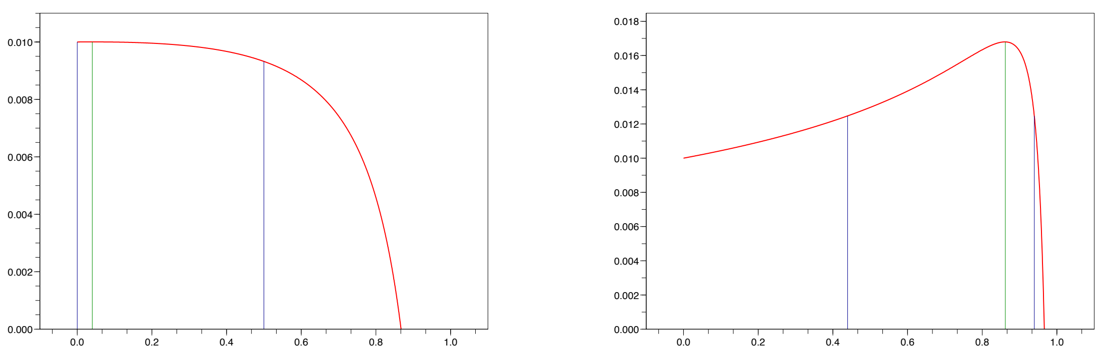

Figure 5: Loss functions for the transitions from the initial equity parameters $\varepsilon_0 = 0.50$ to $\varepsilon_1 = 0.49$ (left) and from $\varepsilon_0 = 0.30$ to $\varepsilon_1 = 0.29$ (right). Neither loss function is monotone which is barely visible in the left case. Yet, the cheapest coalition is connected in the left case with $C = (.5, .8674)$ while it is disconnected in the right case with $C = (0, .4395) \cup (.9395, .9663)$.

remaining gains are distributed. This will be understood as position change. Position changes must not be confused with changes of the overall income distribution. Position changes may occur even if the Lorenz curve remains the same.

In particular, the median income after compensation but before distribution of remaining gains can be lower than the original median income. This decrease and position changes of more than half of all income holders are shown in figure 6.

A bound for the increase in inequality does not seem to exist from the assumptions made so far. Additional considerations or constraints must come into play. One such constraint is that no individual with median income or more before redistribution shall suffer from the inequality increase. Then bifurcation serves as bound for the inequality increase; the equity parameter will never be decreased below the partition function. For example, the strong inequality increase shown in figure 6. were infeasible under this constraint since $pf(0.69) = 0.2375 > 0.12$.

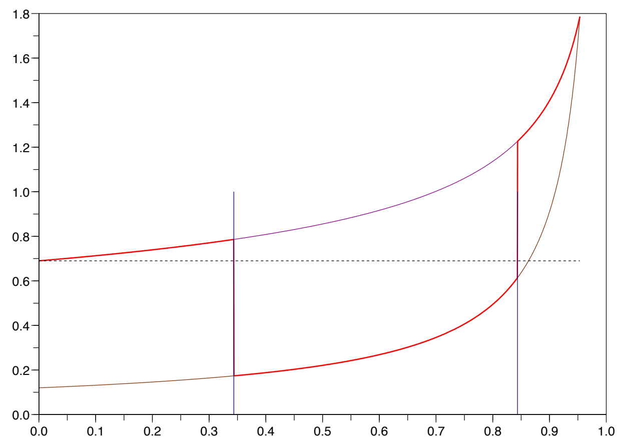

Figure 6: Densities of equity parameter reduction from $\varepsilon_0 = .69$ to $\varepsilon_1 = .12$ shown over the loser segment. All cheapest coalition incomes are compensated to their original levels resulting in all non-coalition incomes (between vertical lines) being decreased below the lowest original income (horizontal line).

If inequality increase were performed along small increments, the equity parameters would not drop significantly below the bifurcation level. More precisely, they would never drop below the bifurcation level minus the increment size. This bound ensures that the lowest incomes before redistribution will not be lowered without halt though the interests of the lowest income holders are overruled by the majority vote.

3.5 Situation after complete redistribution

3.5.1 Proportional redistribution

After compensation and redistribution of the remaining gain, the Lorenz curve will have changed as to mere compensation and income positions may change even more than after mere compensation. One possible distribution scheme for the gain remaining after compensation is that of equal relative increase. All winners and all coalition members receive the same relative increase compared to their original income levels. Formally, increases then are $f_{\varepsilon_0}(x) \to f_{\varepsilon_0}(x) \cdot c$. The relative increase is simply computable from the normalization equation

$$\int_{a_{Int}}^{b_{Int}} f_{\varepsilon_1}(x)dx + \int_0^{a_{Int}} c \cdot f_{\varepsilon_0}(x)dx + \int_{b_{Int}}^1 c \cdot f_{\varepsilon_0}(x)dx = 1.$$

Compensation and distribution of the remaining gains lead to a new Lorenz curve which is not smooth, in general, since its density may have jumps. As to make it comparable to the original distribution, the new Lorenz curve can be approximated by Pareto distributions so that these curves have identical Gini indices. Such a situation is depicted in figure 7. In that example, the income of all original winners and all cheapest coalition partners, after complete redistribution, exceeds the average income.

When the original equity parameter is fixed above the bifurcation level and the reduced equity parameters are decreased but only so little that cheapest coalitions remain connected, then the Lorenz curves after compensation and proportional redistribution are comparable. The Lorenz curve of a smaller reduced equity parameter then Lorenz-dominates the curve for a larger reduced equity parameter. But this in no longer so when the cheapest coalition becomes disconnected; Lorenz curves then may become incomparable.

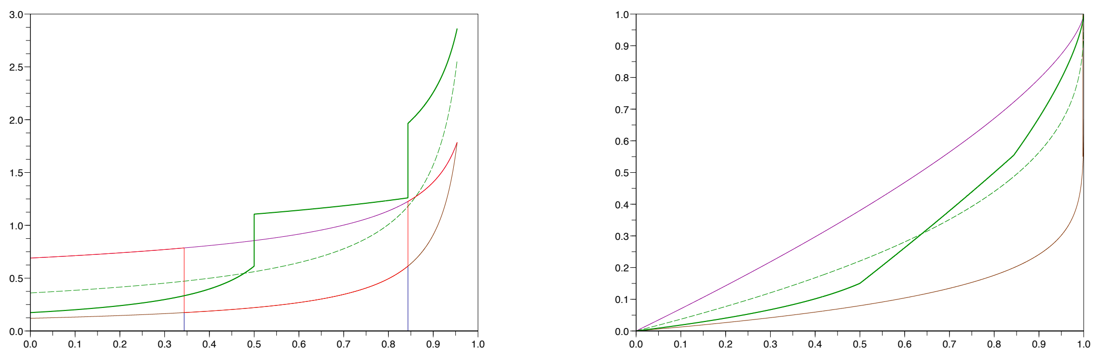

Figure 7: Pareto distributions with Lorenz curves (right) and densities over the loser segment (left) for the original equity parameter 0.69 and reduced equity parameter 0.12 with proportionality value for distribution of remaining gains being $c = 1.60$. The Lorenz curve after complete compensation is not smooth due to the density jumps. Its approximation by a Pareto distribution with same Gini index has equity parameter 0.36 (dashed curves left and right).

It is possible by proportional redistribution to raise the median income above average level with the cheapest coalition remaining connected only if the original equity parameter exceeds a value around 0.62. The reduced equity parameter must then lie around 0.31. Of course, the median income is always raised above its original level if the cheapest coalition remains connected. If not, the median income after compensation and complete proportional redistribution can be smaller than the original median income. The beholder of the median income, the median voter, becomes another individual in such a case. Sample parameters therefore are $\varepsilon_0 = 0.60$ and $\varepsilon_1 = 0.27$.

3.5.2 Inequality bounds

Bounds on distributional inequality can be obtained from a stability notion according to Grandmont [Gra]. This quite technical concept considers an income distribution as stable if an arbitrary objection-distribution that is favored by some objection-majority can be opposed by yet another, suitable distribution proposed some counter-majority.

The counter-majority with its favored counter-distribution must satisfy two essential conditions. First, it must comprise all losers from the objection-majority and offer them at least their current income. Second, it must offer an even higher income than the objection-distribution to all individuals from the intersection of both majorities. As a consequence, this is claimed to create a tie between objection-majority and counter-majority and the tie is claimed to remain unbroken. Thus, the current distribution is left unchanged.

The class of all stable distributions is characterized by their Lorenz curve not intersecting with a certain test function. This boundary condition applies to all distribution types and the test function together with a Pareto Lorenz curve denoting the boundary of the stability region for the Pareto distributions are shown in figure 8. Noteworthy, the Pareto Lorenz curve for the bifurcation parameter lies in the stability region.

3.6 Varying majority levels

Requiring a majority to reach another level than 50% of the votes affects the foregoing quantitative results but not the qualitative results. The gain of the original winners from redistribution is always large enough to compensate required coalition partners. Still, the cheapest coalition is either connected or disconnected depending on the level of inequality increase.

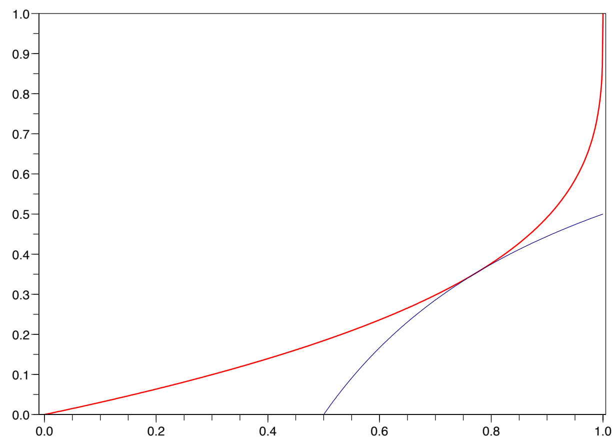

Figure 8: Pareto Lorenz curve with equity parameter $\varepsilon = 0.294$ (upper curve) touching the test curve $y(x) = 1 - 1/(2x)$ with $0.5 \leq x \leq 1$ (lower curve). The region of stable income distributions of the Pareto type lies above the given curve.

Also, bifurcation keeps occuring though values of the bifurcation parameter change. Bifurcation parameters as function of the required majority level α in % are defined as

$$\varepsilon_{bif}(\alpha) = argmax_{\varepsilon \in (0,1]}\ f_\varepsilon(\frac{100-\alpha}{100}) - f_\varepsilon(0).$$

These parameters increase in the required majority levels though they never exceed the value one half, see figure 9. Bifurcation can even be shown to occur for majority levels less than 50% though it sounds paradox to call such situations majority.

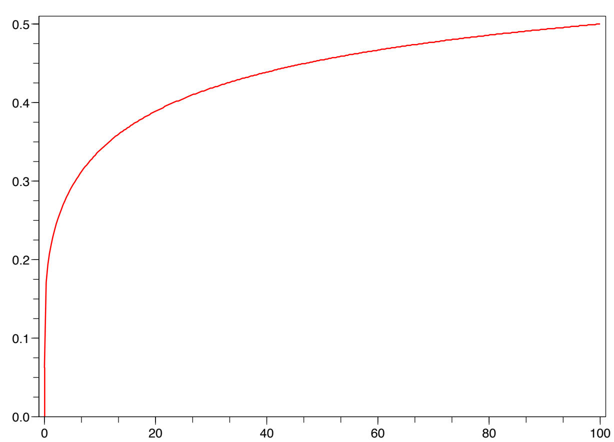

Figure 9: Bifurcation values of the equity parameter as function of the required majority size in %. The foregoing, fixed bifurcation value ε_{bif} lies at $\alpha = 50$.

4 Other income distributions

4.1 One-parametric Lorenz curves

Bifurcation occurs for other income distributions in the same way as for the Pareto distribution and it is verified via the loss function and via the med-min function. Under suitable conditions, which are fulfilled in a variety of cases, differentiable Lorenz curves $F_\vartheta(x)$ with parameter set $\Theta \subseteq I\!R$ have the bifurcation parameter

$$\vartheta_{bif} = argmax_{\vartheta \in \Theta}\, F'_\vartheta(0.5) - F'_\vartheta(0).$$

But for parametric classes of Lorenz curves, in general, the med-min function neither is continuous nor need it have two monotonicity segments. These two complications are addressed in the appendix. A single monotonicity segment only may indicate the absence of a bifurcation parameter.

One issue to be investigated is whether a candidate value for the bifurcation parameter can be guaranteed to exist. This depends on the parameter set being sufficiently large. The cheapest coalition will always be disconnected for a suitable increase in inequality if the Lorenz curves approximate the zero line arbitrarily close meaning that $\lim_{\vartheta \to \vartheta_0} F_\vartheta(x) = 0$ for all $x \in [0, 1)$ with $\vartheta_0 \notin \Theta$ admitted. The limit condition together with convexity of all Lorenz curves implies $\lim_{\vartheta \to \vartheta_0} F'_\vartheta(0.5) = \lim_{\vartheta \to \vartheta_0} F'_\vartheta(0) = 0$ from which disconnection can be deduced.

Bifurcation parameters for standard one-parametric classes of Lorenz curves are summarized in table 1; classes are specified without possible overlaps. Out of two Lorenz curves with distinct parameters, one curve Lorenz-dominates the other within each class. Med-min functions for two one-parametric distributions are sketched in figure 10.

Lorenz curve family Function	Parameter range	Name		Bifurcation parameter
$F_\varepsilon(x) = 1 - (1-x)^\varepsilon$	$0 < \varepsilon \leq 1$	Pareto	$\downarrow$	0.45433
$F_a(x) = x^a$	$a \geq 1$	polynomial	$\uparrow$	$1.4428 = 1/\log_e 2$
$F_{\varepsilon,2}(x) = 0.5 \cdot x^{1/\varepsilon} + 0.5 \cdot (1 - (1-x)^\varepsilon)$	$0 < \varepsilon \leq 1$	convex	$\downarrow$	0.41673
$F_g(x) = x \cdot g^{x-1}$	$g \geq 1$	Gupta	$\uparrow$	11.48
$F_m(x) = \frac{m\,x}{1+(m-1)\,x}$	$0 < m \leq 1$	mixed	$\downarrow$	0.3646
$F_\vartheta(x) = \frac{(1-\vartheta)^2\,x}{(1+\vartheta)^2 - 4\,\vartheta\,x}$	$0 \leq \vartheta < 1$	Aggarwal	$\uparrow$	0.2956
$F_\beta(x) = x \cdot \frac{\beta-1}{\beta-x}$	$\beta > 1$	Rohde	$\uparrow$	1.5741
$F_\kappa(x) = \frac{e^{\kappa\,x}-1}{e^\kappa - 1}$	$\kappa > 0$	Chotikapanich	$\uparrow$	2.5566

Table 1. One-parametric Lorenz curves and their bifurcation parameters. Up-arrows indicate that unevenness increases in the parameter, down-arrows indicate the opposite.

All foregoing one-parametric income distributions have bifurcation parameters that lie in the interior of the parameter set. But this is not always so. Examples of bifurcation parameters at boundaries of the parameter set can be constructed from two-parametric distributions, see below, by coupling both parameters. Then, the coupled product distribution and the coupled Rasche distribution both have the same parameter set as the Pareto distribution and both have bifurcation parameter at the upper bound of the parameter set $(0, 1]$. The respective Lorenz curves are

$$\begin{aligned} F(x) &= x^{1/\varepsilon} \cdot (1 - (1-x)^\varepsilon) \\ F(x) &= (1 - (1-x)^\varepsilon)^{1/\varepsilon}. \end{aligned}$$

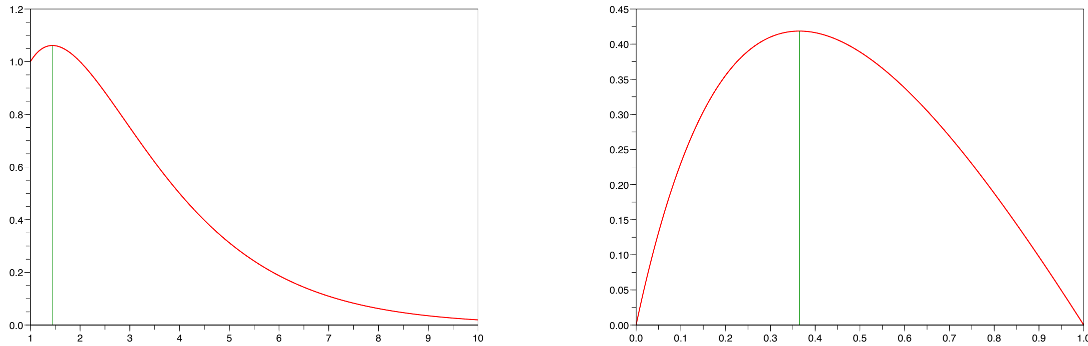

Figure 10: Med-min function for the polynomial Lorenz curves (left) and the mixed Lorenz curves (right). Unevenness increases in the polynomial case and decreases in the mixed case with the parameter. Bifurcation parameters are indicated by vertical lines; values as in table 1, rightmost column, rows two and five.

Both bifurcation parameters $\varepsilon_{bif} = 1$ belonging to the Egalitarian distribution implies that both med-min functions are discontinuous at the upper interval boundary. So, every (small) inequality increase entails disconnection of the cheapest coalition – independent from the original inequality level as long as the original distribution is not Egalitarian. This corresponds to the partition function being the identity function for all parameters below one.

Though the med-min functions exhibit a single maximum or supremum for the one-parametric Lorenz curves given so far, examples with multiple maxima exist. These, however, do not turn up as standard classes of one-parametric Lorenz curves but can be specifically constructed. Inequality increase from a fixed, original level may then warrant the cheapest coalition to be, alternatingly, disconnected, connected etc. depending on the new level of inequality.

All standard Lorenz curves from table 1 lie in the stability region from section 3.5.2 for their bifurcation parameters. Thus, starting with the identical distribution or from close to there and increasing inequality, bifurcation occurs before instability. Sample values of parameters which bound the stability region are $\varepsilon_{GR} = 0.294$ for the Pareto distribution, $a_{GR} = 3.4$ for the polynomial distribution, $m_{GR} = 0.17$ for the mixed distribution and $\kappa_{GR} = 3.99$ for the Chotikapanich distribution; comp. figure 8. It is conjectured that all bifurcation parameters of one-parametric distributions belong to stable distributions.

4.2 Two-parametric Lorenz curves

Disconnection of the cheapest coalition can be observed for two-parametric Lorenz curves as for one-parametric Lorenz curves, yet constellations may be more complicated. Some standard two-parametric Lorenz curves are given in table 2.

Lorenz curve family			
Function	Parameter ranges	Name	
$F_{\varepsilon,a}(x) = x^a\,(1-(1-x)^{\varepsilon})$	$0 < \varepsilon \leq 1$, $a \geq 0$	product or Ortega	$\downarrow\uparrow$
$F^R_{\varepsilon,a}(x) = (1-(1-x)^{\varepsilon})^a$	$0 < \varepsilon \leq 1$, $a \geq 1$	Rasche	$\downarrow\uparrow$

Table 2. Selected two-parametric Lorenz curves with, again, up-arrows indicating increasing unevenness in the parameter and down-arrows indicating the opposite. Both classes comprise the one-parametric Pareto distribution; the first with $a = 0$ and the second with $a = 1$.

In case the cheapest coalition is disconnected, the strongest loser segment lies at the lower interval bound for the one-parametric distributions from section 4.1. This need not be so for the loss function of the product Lorenz curve as indicated by figure 11. Strongest loser segments need not begin at zero. Note that the loss function for multi-parametric Lorenz curves is understood in the same way as for one-parametric Lorenz curves, namely as $l(x) = F'_{\varepsilon_0,a_0}(x) - F'_{\varepsilon_1,a_1}(x)$ etc.

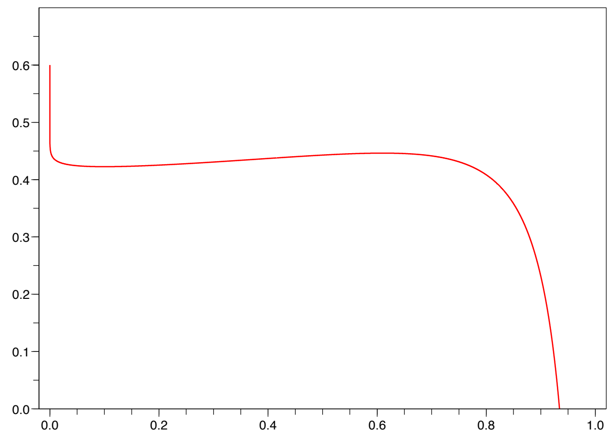

Figure 11: The loss function for the two-parametric product distribution with initial parameter pair $(\varepsilon_0, a_0) = (0.6, 0)$ changed to $(\varepsilon_1, a_1) = (0.2, 0.05)$ has three monotonicity segments. As a consequence of the loss being maximal at zero, the cheapest coalition does not begin there. But strong losers are actually separated from mild losers because the loss function has a local maximum above 0.5.

Connectivity of the cheapest coalition is characterized by the loss for each above median income not exceeding the loss for any below median income

$$\text{Connectivity} \Longleftrightarrow l(x) \geq l(0.5) \geq l(z) \ \forall\, x \in [0, 0.5) \text{ and } \forall\, z \in (0.5, 1).$$

Since the condition $l(0.5) > l(0)$ is only sufficient for disconnection of the cheapest coalition, the maximum point of the med-min function can only indicate parameter constellations that lead to disconnection. Yet there may be additional constellations of disconnection as in figure 11. The range of parameters for which disconnection of the cheapest coalition is guaranteed according to the comparison of median and minimum income is not of a simple form like an interval as indicated by figure 12.

The shape of the contourline suggests that there a no strict two-parametric counterpart to the bifurcation parameter from the one-parametric case. However, for each one parameter, there is a value of the other parameter such that incremental changes result in disconnection of the cheapest coalition. More precisely, for every ε_0 there is a value a_0 such that all parameter pairs (ε_1, a_1) with $\varepsilon_1 \leq \varepsilon_0$ and $a_1 \geq a_0$ result in disconnection when at least one inequality is strict. This functional dependency is denoted by the bifurcation function. It is obtained by maximizing the med-min function over a for given ε

$$a(\varepsilon) = argmax_a \, F'_{\varepsilon,a}(0.5) - F'_{\varepsilon,a}(0).$$

Similarly, the roles of the parameters can be interchanged. Via critical point conditions for med-min functions, bifurcation functions can be computed for the product Lorenz curves and the Rasche curves explicitly as

$$a(\varepsilon) \quad = \quad \frac{0.5^{\varepsilon} - 1 - \varepsilon \cdot 0.5^{\varepsilon} \cdot \log_e 0.5}{(1 - 0.5^{\varepsilon}) \cdot \log_e 0.5}$$

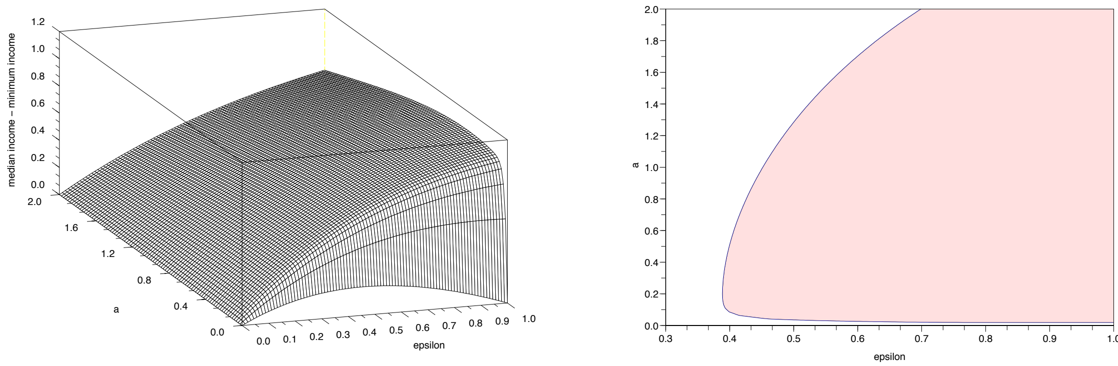

Figure 12: Surface (left) and contourline for level 0.6 (right) of the med-min function of the product Lorenz curves. All function values in the unshaded area are smaller than in the shaded area so that each incremental move from the shaded area (including the contourline) into the unshaded area causes disconnection of the cheapest coalition.

$$a^R(\varepsilon) \quad = \quad \frac{-1}{\log_e(1-0.5^{\varepsilon})}.$$

Since the bifurcation function gives maximum values when one parameter is allowed to vary freely, it can be considered as ridge function of the med-min function landscape. The bifurcation function of the product Lorenz curve is increasing, concave and yet almost linear as shown in figure 13 (right).

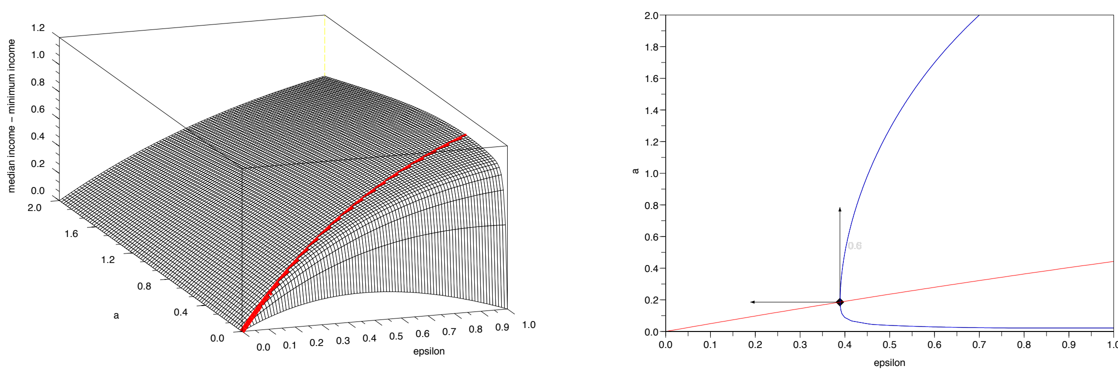

Figure 13: Contourline for level 0.6 (right) as in figure 12, bifurcation function (right) and bifurcation function lifted to the med-min surface (left) illustrating the name ridge function. All differences between median and minimum income in the left upper quadrant from the leftmost point on the contourline are smaller than at the leftmost point. Each incremental move from this so-called quadrant point into the quadrant causes disconnection of the cheapest coalition.

4.3 Three-parametric and other Lorenz curves

Three- and more-parametric Lorenz curves are analyzed in a way analogous to two-parametric curves with bifurcation functions now being allowed to have two and more independent variables. These give rise to bifurcation manifolds. Avoidance of repetitions calls for these not being detailed here. Some prominent three-parametric Lorenz curves are compiled in table 3.

Function	Lorenz curve family Parameter ranges	Name	
$F(x) = x - a \cdot x^{\delta} \cdot (1-x)^{\beta}$	$a > 0,\ 0 < \delta \leq 1,\ 0 < \beta \leq 1$	Kakwani or Beta	$\uparrow\downarrow\downarrow$
$F(x) = x^{\alpha} \cdot (1 - (1-x)^{\varepsilon} e^{\beta} x)$	$0 < \varepsilon \leq 1,\ \alpha \geq 0,\ 0 \leq \beta \leq \varepsilon$	Abdalla and Hassan	$\downarrow\uparrow\uparrow$
$F(x) = x^{\alpha} \cdot (1 - (1-x)^{\varepsilon})^{\beta}$	$0 < \varepsilon \leq 1,\ \alpha \geq 0,\ 1 \leq \beta$	General Pareto	$\downarrow\uparrow\uparrow$

Table 3. Selected three-parametric Lorenz curves with up-arrows and down-arrows as before.

Semi-parametric and non-parametric Lorenz curves are not considered here since their scope is high quality fit to empirical income data and robustness to uncertainty in these data rather than anything else. Complicated functional forms and partial indefiniteness make it difficult to go beyond the descriptive with these kinds of Lorenz curves.

5 Conclusion

Least-cost majorities for income redistribution have been shown to be either connected or disconnected depending on initial and target distributions. It seems that the combination of an economical perspective and a rudimentary democracy model can lead to arbitrary inequality increase. A price to pay is that income orders may change drastically in the lower and medium income ranges.

Results should support the formation of more complex models including growth and games like extensions of ultimatum bargaining that consider least-cost coalitions.

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Appendix: Bifurcation parameters for one-parametric Lorenz curves – general case

In general, the med-min function neither is continuous nor need it have two monotonicity segments for parametric classes of Lorenz curves. The first of these complications is addressed by replacing the maximum with the supremum and the second is taken care of by modifying the parameter set. When the modified parameter set is empty, a bifurcation parameter does not exist. The bifurcation parameter and the modified parameter set are stated as

$$\begin{aligned}\vartheta_{bif} &= argsup_{\vartheta\in\Theta_=}\, F'_\vartheta(0.5) - F'_\vartheta(0)\\ \Theta_= &= \{\vartheta \in \Theta \| \,\exists\, \vartheta^* \in \Theta - \{\vartheta\} \text{ with } F_{\vartheta^*} \text{ Lorenz-dominating } F_\vartheta \text{ and}\\ &\qquad\qquad F'_{\vartheta^*}(0.5) - F'_{\vartheta^*}(0) \le F'_\vartheta(0.5) - F'_\vartheta(0)\}.\end{aligned}$$

The supremum-argument is understood as argument leading to the maximum, if the maximum exists or, else, as

$$\vartheta_{bif} = \lim_{i\to\infty} \vartheta_i \text{ with } \lim_{i\to\infty} F'_{\vartheta_i}(0.5) - F'_{\vartheta_i}(0) = sup_{\vartheta\in\Theta}\, F'_\vartheta(0.5) - F'_\vartheta(0).$$

The supremum over the differences between median income and minimum income always exists since the median income is bounded by twice the average income. Thus, $F'_\vartheta(0.5) - F'_\vartheta(0) \le 2$ for all $\vartheta \in \Theta$. An alternative condition for a value ϑ_{bif} qualifying as bifurcation value is that for all $\vartheta \in \Theta$ with $\vartheta \neq \vartheta_{bif}$ exists $\vartheta^* \in \Theta$ between ϑ and ϑ_{bif} such that $F'_{\vartheta^*}(0.5) - F'_{\vartheta^*}(0) > F'_\vartheta(0.5) - F'_\vartheta(0)$.