On duality transformations between Lorenz curves and distribution functions

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1 Introduction

The relation between distribution functions and Lorenz curves is often denoted as their duality. Duality is established via transformations which can be understood as forward and backward transformations. It is often implicated that the interplay between distribution functions and Lorenz curves depends on the functions alone. It is the purpose of this short investigation to show that several transformations exist – at least in special cases.

Formally, a distribution function or probability distribution function is a non-decreasing, right-continuous function F with $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$. The last condition will always be satisfied here as considerations are restricted to distribution functions which have no mass over negative reals so that $F(0-) = 0$. Moreover, all distribution functions considered have finite expectation.

A Lorenz curve is any non-decreasing convex function L with $L(0) = 0$ and $L(1) = 1$. It should be noted that each Lorenz curve already is a distribution function as considered here and each Lorenz curve is invertible over its whole domain if and only if it has no constant segment at level zero.

2 Transformations

2.1 Setting

The transformation

$$L(x) = H(F)(x) = \frac{\int_0^x F^{-1}(u)\,du}{\int_0^1 F^{-1}(u)\,du},\ 0 \le x \le 1$$

is known to lead from any distribution function to a Lorenz curve when the generalized inverse $F^{-1}(u) = \inf\{w|\, F(w) \ge u\}$ is used in case the distribution function is not invertible, see [Tho]. Every Lorenz curve as defined by the properties from the introduction has at least one distribution function whose transform is the given Lorenz curve [IK]. The opposing transformation

$$F(x) = K(L)(x) = L'(x)$$

is specified for all Lorenz curves, if the generalized derivative $L'(x) = \sup\{w|\, L'^{-}(w) \le x\}$ is used whenever the Lorenz curve is not differentiable in x. The generalized inverse is the left-sided derivative which exists for all Lorenz curves for all strictly positive arguments due to monotonicity. The transformation and its opposite have often been considered. Extensions to so-called generalized Lorenz curves, which differ from ordinary Lorenz curves by the absence of normalization, are given in [Thi].

The transformation and its opposite are not inverse to each other, meanimg that $K(H(F))(x) \neq F(x)$ is possible. Yet, $H(K(H(F)))(x) = H(F)(x)$ for all distribution functions F and all $x \in [0,1]$.

2.2 Fixed points

The transformation can be applied, in particular, to Lorenz curves since these are special distribution functions. A sequence of successive transforms is given in table 1 and a fixed point of the transformation is the power function $L(x) = x^{a_1}$ with golden section ratio $a_1 = (1+\sqrt{5})/2 = 1.61803398....$ A derivation of the fixed point is obtained from inversion and integration preserving the power type of a function. Considering a Lorenz curve $L(x) = x^a$ with $a \geq 1$ and inverse $L^{-1}(x) = x^{1/a}$ results in the fixed point equations

$$\begin{aligned} x^a &= \frac{\int_0^x u^{1/a}\,du}{\int_0^1 u^{1/a}\,du} \\ &= \frac{\frac{1}{1/a+1}\cdot u^{1/a+1}|_0^x}{\frac{1}{1/a+1}\cdot u^{1/a+1}|_0^1} \\ &= x^{1/a+1}. \end{aligned}$$

Comparison of exponents results in

$$\begin{aligned} a = \frac{1}{a}+1 \quad &\Longleftrightarrow \quad a^2 - a - 1 = 0 \\ &\Longleftrightarrow \quad a = \frac{1}{2} \pm \sqrt{\frac{1}{4}+1} \\ &\Longleftrightarrow \quad a = \frac{1\pm\sqrt{5}}{2}. \end{aligned}$$

The golden section ratio $a = a_1$ indeed leads to a Lorenz curve since the exponent is greater than one ensuring convexity of the power function.

Arbitrary Lorenz curves allow to compute their formal expectation value $EW = \int_0^1 L^{-1}(w)\,dw$. This value is, however, more a dispersion measure than a measure of central tendency as is typical for expectations. The reason for the present case is that the formal expectation of any Lorenz curve is in 1:1 relation to its Giniindex

$$EW = \frac{1}{2} + \frac{1}{2}\cdot \text{Giniindex}$$

with the Giniindex of any Lorenz curve being Giniindex $= 2\cdot \int_0^1 x - L(x)\,dx$. The fixed point satisfies the equations

$$\begin{aligned} \frac{1}{EW} &= a_1 \\ \frac{EW}{1-EW} &= a_1, \end{aligned}$$

which means that the area of the larger segment has the same proportion of the unit square as the smaller segment has of the larger, see figure 1 (left). The fixed point is contracting so that the sequence of successive transforms

$$L_{n+1}(x) = \frac{\int_0^x L_n^{-1}(u)\,du}{\int_0^1 L_n^{-1}(u)\,du}$$

converges towards the fixed point for all initial Lorenz curves $L_0(x) = x^a$. The sequence for $a = 1$ is given in table 1. It is yet unknown if the fixed point is contracting from other initial Lorenz curves. Anyway, every differentiable fixed point satisfies the ordinary differential equation

$$L'(L(x)) = c \cdot x \Longleftrightarrow L'(x) = c \cdot L^{-1}(x) \Longleftrightarrow y' = c \cdot y^{-1} \text{ with } c > 0.$$

Validity of the ODE is easily verified by using the original fixed point equation with $x = L(z)$ and renaming the independent variable z again as x. It is yet unknown if this simple-looking differential equation admits other solutions than the power function with golden ratio parameter.

3 Alternative transformation

Each invertible Lorenz curve yields another Lorenz curve by reflection along the decreasing diagonal $1-x$. The reflected Lorenz curve is computed as

$$L^{ref}(x) = 1 - L^{-1}(1 - x).$$

The reflection of the reflection is the original Lorenz curve and a Lorenz curve may coincide with its reflected curve like $L(x) = 1 - \sqrt{1 - x^2} = L^{ref}(x)$ and $L_m(x) = \frac{m\,x}{1+(m-1)\,x} = L_m^{ref}(x)$ for $0 < m \leq 1$.

A distribution function may result in two distinct Lorenz curves by applying transformation H and then reflecting the resulting Lorenz curve. This raises the issue of obtaining the reflected curve without explicit reflection operation. The following transformation should achieve this.

3.1 Setting

An alternative to the foregoing transformation can be established for distribution functions with compact support. By scaling and shifting it can be assumed, without loss of generality, that such a distribution function satisfies $F(1) = 1$. Making use of the survival function $\bar{F}(x) = 1 - F(x)$ and defining the survival Lorenz curve $\bar{L}(x) = 1 - L(x)$ allows to consider the transformation

$$\bar{L}^{-1}(x) = \frac{\int_0^{1-x} \bar{F}(u)\,du}{\int_0^1 \bar{F}(u)\,du},\ 0 \leq x \leq 1.$$

A Lorenz curve is obtained by

$$L(x) = H_a(F)(x) = 1 - \varphi^{-1}(x) \text{ for } \varphi(x) = f(1 - x) \text{ where } f(x) = \frac{\int_0^x \bar{F}(u)\,du}{\int_0^1 \bar{F}(u)\,du}.$$

The function $f(x)$ is increasing and concave. This implies that $f(1 - x)$ is decreasing and concave so that $\varphi^{-1}(x)$ is decreasing and concave so that, in turn, $1 - \varphi^{-1}(x)$ is increasing and convex. For the transformation being "practically" applicable it is, of course, necessary that the inversion can be computed in closed form.

Iteration	H	H_a	Giniindex
1	x		0
2	x^2	$1-(1-x)^{1/2}$	$1/3$
3	$x^{3/2}$	$1-(1-x)^{2/3}$	$1/5$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
limit	x^{a_1}	$1-(1-x)^{1/a_1}$	$\sqrt{5}-2=0.2361$

Table 1: Two converging sequences of Lorenz curves whose Giniindices are identical in each iteration and in the limit since they are refections of each other. Exponents of all iterates are ratios of successive Fibonacci numbers.

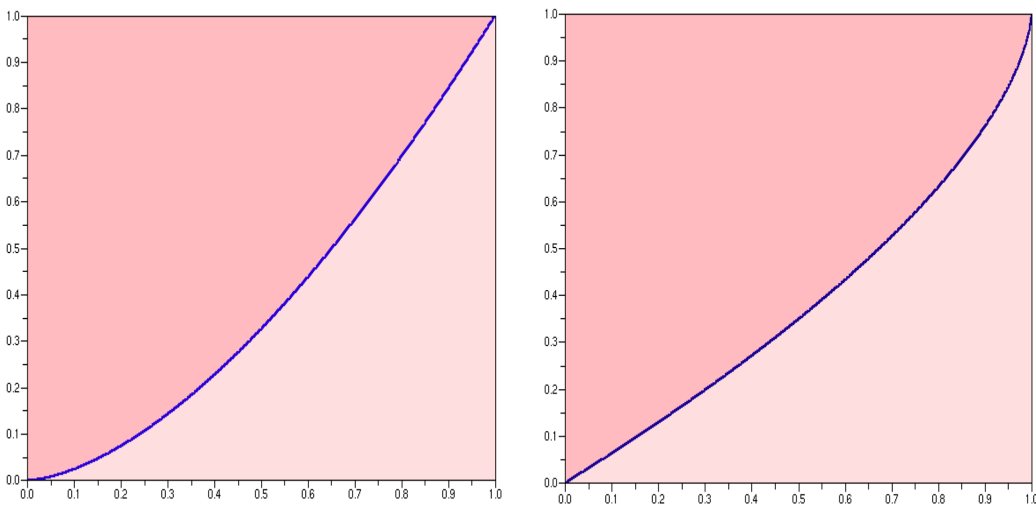

Figure 1: Lorenz curve of power type with golden ratio parameter (left) and of Pareto type with inverse golden ratio parameter (right). Both Lorenz curves are reflections of each other and, thus, have the same Giniindex.

3.2 Fixed points

In principle, the alternative transformation can be invoked for any Lorenz curve and it has the fixed point $L(x) = 1-(1-x)^{1/a_1}$. This is the Pareto-Lorenz curve whose parameter equals the inverse of the golden section ratio $\varepsilon = 1/a_1 = (\sqrt{5}-1)/2 = 0.61803398....$ The difference between the original and the alternative transformation is illustrated for a common initial distribution function in table 1 with limit case shown in figure 1.

Areas related to the formal expectation EW satisfies the same proportions for this fixed point as for the fixed point of the ordinary transformation, see figure 1 (right).

$$\begin{aligned}\frac{1}{EW} &= a_1 \\ \frac{EW}{1-EW} &= a_1.\end{aligned}$$

In the same way as for the original transformation, it is yet unclear whether the alternative transformation has additional fixed points.

References

[IK] Iritani, J., Kuga, K., "Duality between the Lorenz curves and the income distribution functions", Economic Studies Quarterly 34, 1983, p. 9-21.

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[Tho] Thompson, W.A., "Fisherman's luck", Biometrics 32, 1976, p. 265-271.